Ricci Curvature Bounds for Warped Products

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Abstract

We prove generalized lower Ricci curvature bounds for warped products over complete Finsler manifolds. On the one hand our result covers a theorem of Bacher and Sturm concerning euclidean and spherical cones ([3]). On the other hand it can be seen in analogy to a result of Bishop and Alexander in the setting of Alexandrov spaces with curvature bounded from below ([1]). For the proof we combine techniques developed in these papers. Because the Finslerian product metric can degenerate we regard a warped product as metric measure space that is in general neither a Finsler manifold nor an Alexandrov space again but a space satisfying a curvature-dimension condition in the sense of Lott-Villani/Sturm.
Keywords: warped product, curvature-dimension condition, Finsler manifold, Alexandrov space

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1 Introduction

By using displacement convexity of certain entropy functionals on the L^2-Wasserstein space Sturm in [24, 25] and Lott-Villani in [12] independently introduced a synthetic notion of generalized lower Ricci curvature bound for metric measure spaces (X, d_X, m_X). Combined with an upper bound $N \in [1, \infty)$ for the dimension that leads to the so-called curvature-dimension condition $CD(K, N)$. For complete weighted Riemannian manifolds this is equivalent to having bounded N-Ricci curvature in the sense of Bakry and Emery. The strength of this approach comes from the stability under Gromov-Hausdorff convergence and the numerous corollaries like Brunn-Minkowski inequality, Bishop-Gromov volume growth and eigenvalue estimates that now hold in a very general setting.

The other object we deal with is the concept of warped product $B \times_f F$ between two metric spaces (B, d_B) and (F, d_F) and a Lipschitz function $f : B \to [0, \infty)$. It is a generalization of the cartesian product and well-known examples are euclidean or spherical cones, where the first factor is $[0, \infty)$ or $[0, \pi]$ respectively and the warping functions are r or $\sin r$. These cones appear naturally as tagent cones of other spaces. But also more complicated warped product constructions

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are important and can be tools that allow to prove results beside the warped product itself (see for example [11] and corollary (3.19)).

The aim of this paper is to draw some connections between warped products and the curvature-dimension condition when the underlying spaces are weighted Finsler manifolds. More precisely, we want to deduce a curvature-dimension bound for $B\times_f F$ provided suitable conditions for B, F and f. On the one hand such a result would be interesting because examples that mainly inspired the definition of curvature-dimension in the sense of Lott-Villani/Sturm came from measured Gromov-Hausdorff limits of Riemannian manifolds. On the other hand we already have stability of CD under tensorization with respect to the cartesian product. So the stability under the warped product construction would yield a new class of examples and would be another justification for the new approach. The main theorem is

Theorem 1.1. *Let B be a complete, d-dimensional space with curvature bounded from below (CBB) by K in the sense of Alexandrov such that $B\backslash\partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K$-concave and smooth on $B\backslash\partial B$. Assume $\partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N \geq 1$ and $K_F \in \mathbb{R}$. If $N = 1$ and $K_F > 0$, we assume that $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. In any case F satisfies $CD((N-1)K_F, N)$ where $K_F \in \mathbb{R}$ such that*

1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $|\nabla f|_p \leq \sqrt{K_F}$ for all $p \in \partial B$.*

Then the N-warped product $B \times_f^N F$ satisfies $CD((N + d - 1)K, N + d)$.

A continuous function $f : B \to \mathbb{R}_{\geq 0}$ is said to be $\mathcal{F}K$-concave if its restriction to every unit-speed geodesic satisfies $u'' + Ku \leq 0$ in the barrier sense. For a Riemannian manifold and smooth f that is equivalent to $\nabla^2 f(\cdot) \leq -Kf|\cdot|^2$ where $\nabla^2 f$ denotes the Hessian. The boundary of B is defined in the sense of an Alexandrov space. The notion of N-warped product is a slight generalisation of the usual warped product where $N \geq 1$ is an additional parameter that describes a dimension bound for F (Definition 2.22). We only consider smooth ingredients, so also the measure m_F is assumed to be smooth (see the discussion after Definition 2.16).

Corollary 1.2. *Let B be a complete, d-dimensional space with CBB by K such that $B\backslash\partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K$-concave and smooth on $B\backslash\partial B$. Assume $\emptyset \neq \partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N > 1$. Then the following statements are equivalent*

(i) (F, m_F) satisfies $CD((N-1)K_F, N)$ with $K_F \geq 0$ and

$$|\nabla f|_p \leq \sqrt{K_F} \text{ for all } p \in \partial B.$$

(ii) The N-warped product $B \times_f^N F$ satisfies $CD((N + d - 1)K, N + d)$

Why could we expect such a relation? To answer this question let us consider first the special case where f doesn't vanish and B, F and $B \times_f F$ are Riemannian manifolds. The formula for the Ricci-tensor of warped products is ([17])

$$\operatorname{ric}_{B\times_f F}(\tilde{X}_{(p,x)} + \tilde{V}_{(p,x)}) = \operatorname{ric}_B(X_p) - n\frac{\nabla^2 f(X_p)}{f(p)} + \operatorname{ric}_F(V_x) - \left(\frac{\Delta f(p)}{f(p)} + (n-1)\frac{\nabla f_p}{f^2(p)}^2\right)|\tilde{V}_{(p,x)}|^2 \quad (1)$$

where $\tilde{X}$ and $\tilde{V}$ are horizontal and vertical lifts on $B \times_f F$ of vector fields X and V on B and F respectively. For the notion of horizontal and vertical lifts we refer to [17]. Then under the assumptions on B and f in the theorem, we get

$$\operatorname{ric}_F(v, v) \geq (n-1)K_F|v|^2 \implies \operatorname{ric}_{B\times_f F}(\xi + v, \xi + v) \geq (n + d - 1)K|\xi + v|^2 \quad (2)$$

for every $v \in TF$ and for every $\xi + v \in TB \times_f F = TB \oplus TF$ respectively.

But even in the smooth setting one problem still occurs. When we allow the function f to vanish and in most of the interesting cases, that will happen, the metric tensor degenerates and the warped product under consideration is no longer a manifold. Especially we have no notion for the Ricci-tensor at singularity points. One strategy to solve this problem could be to cut the singularities and consider only what's left over. But in general that space neither will be complete nor strictly intrinsic. And even if we don't cut singularities then the Ricci bound on the regular part doesn't need to yield any bound for the curvature-dimension condition on the whole space.

But the warped product together with its distance function and its volume measure is a complete and strictly intrinsic metric measure space, where the curvature-dimension condition in the sense of Lott-Villani/Sturm can be defined without problem. So it is convenient to state a result in terms of that condition and use tools from optimal transportation theory to circumvent the problem that comes from the singularities. A first step in this direction was done by Bacher and Sturm in [3] where they show that the euclidean cone over some Riemannian manifold satisfies $CD(0, N)$ if and only if the underlying space satiesfies $CD(N-1, N)$. Our main theorem is an analog of this in the setting of warped products and Finsler manifolds.

One can see that the curvature bound that holds by formula (1) on the regular part passes to the whole space provided the given assumptions. In general that won't be true. For example consider $N = 1$ and the euclidean cone over $F = \mathbb{R}$. Then equation (1) is still true where f doesn't degenerate, but the cone doesn't satisfy any curvature-dimension condition (see [3]). The main part of the proof is to show that the set of singularity points doesn't affect the optimal transportation of mass and therefore, doesn't affect the convexity of any entropy functional on the L^2-Wasserstein space (under the given assumptions).

Although our main theorem only works when the underlying spaces are Riemannian manifolds and weighted Finsler manifolds respectively the statement that the tranport doesn't see any singularities (see Theorem 3.4) holds even in the setting where one can replace Riemannian and Finsler manifolds with Alexandrov spaces and metric measure spaces respectively and, though we are not able to prove it yet, we conjecture an analog of our theorem in this setting (see 3.15). We also want to mention that there is a very good compatibility of warped products with the concept of generalized lower (and also upper) bound for curvature in the sense of Alexandrov. If we assume B and F to be Alexandrov spaces with CBB by K and K_F respectively, where $\partial B \subset f^{-1}(0)$ and K_F and f as in the theorem then the corresponding warped product has CBB by K_F. This result was proved by Alexander and Bishop in [1] (see also Theorem 2.11).

The organisation of the paper goes as follows. In the next section we give a little introduction to warped products and provide all the results that are important for us. We treat the general case of metric measure spaces but then we concentrate on the more specific situation of our theorem, namely Alexandrov spaces, Riemannian manifolds and Finsler manifolds. The needed results of Alexander and Bishop from [1, 2] are presented. We also give the definition of the modified Bakry-Emery N-Ricci-tensor for Finsler manifolds and the definition of curvature-dimension condition for metric measure spaces. In the third section we prove Theorem 3.8. There we first generalize the formula of the Ricci-tensor for warped products to the setting of weighted Riemannian manifolds and N-Ricci-tensor and to the setting of Finsler manifolds. Then we prove the theorem that states that under our assumptions the optimal transport doesn't see any singularities. We adapt the idea of Bacher and Sturm where the main difference arises in the fact that singularity sets can have dimension bigger than zero. After the proof of the main theorem we add a self-contained presentation for the result of the existence of optimal maps in warped products that can not directly be derived from the classical result of McCann but from more general consideration of Villani in [26].

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2 Preliminaries

2.1 Basic definitions and notations

Throughout this paper, (X, d_X) always will denote a complete separable metric space and m_X a locally finite measure on $(X, \mathcal{B}(X))$ with full support. That is, for all $x \in X$ and all sufficiently small $r > 0$ the volume $m_X(B_r(x))$ of balls centered at x is positive and finite. We assume that X has more than one point.

The metric space (X, d_X) is called a *length space* or *intrinsic* if $d_X(x, y) = \inf \mathrm{L}(\gamma)$ for all $x, y \in X$, where the infimum runs over all curves γ in X connecting x and y. (X, d_X) is called a *geodesic space* or *strictly intrinsic* if every two points $x, y \in X$ are connected by a curve γ with $d_X(x, y) = \mathrm{L}(\gamma)$. Distance minimizing curves of constant speed are called *geodesics* or minimizers. The space of all geodesics $\gamma : [0, 1] \to X$ will be denoted by $\Gamma(X)$. (X, d_X) is called *non-branching* if for every tuple (z, x_0, x_1, x_2) of points in X for which z is a midpoint of x_0 and x_1 as well as of x_0 and x_2, it follows that $x_1 = x_2$. A triple (X, d_X, m_X) with d_X strictly intrinsic will be called *metric measure space.*

$\mathcal{P}_2(X, d_X)$ denotes the L^2*-Wasserstein space* of probability measures μ on $(X, \mathcal{B}(X))$ with finite second moments which means that $\int_X d_X^2(x_0, x)d\mu(x) < \infty$ for some (hence all) $x_0 \in X$. The L^2-*Wasserstein distance* $d_W(\mu_0, \mu_1)$ between two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(X, d_X)$ is defined as

$$d_W(\mu_0, \mu_1) = \inf \left\{ \left(\int_{X \times X} d_X^2(x, y)\, d\pi(x, y) \right)^{1/2} : \pi \text{ coupling of } \mu_0 \text{ and } \mu_1 \right\}.$$

Here the infimum ranges over all *couplings* of μ_0 and μ_1, i.e. over all probability measures on $X \times X$ with marginals μ_0 and μ_1. Equipped with this metric, $\mathcal{P}_2(X, d_X)$ is a complete separable metric space. The subspace of m_X-absolutely continuous measures is denoted by $\mathcal{P}_2(X, d_X, m_X)$.

Definition 2.1. (i) A subset $\Xi \subset X \times X$ is called d_X^2*-cyclically monotone* if and only if for any $k \in \mathbb{N}$ and for any family $(x_1, y_1), \ldots, (x_k, y_k)$ of points in Ξ the inequality

$$\sum_{i=1}^{k} d_X^2(x_i, y_i) \le \sum_{i=1}^{k} d_X^2(x_i, y_{i+1})$$

holds with the convention $y_{k+1} = y_1$.

(ii) Given probability measures μ_0, μ_1 on X, a probability measure π on $X \times X$ is called *optimal coupling* of them iff π has marginals μ_0 and μ_1 and

$$d_W(\mu_0, \mu_1) = \int_{X \times X} d_X^2(x, y)\, d\pi(x, y).$$

(iii) A probability measure Π on $\Gamma(X)$ is called *dynamical optimal transference plan* iff the probability measure $(e_0, e_1)_*\Pi$ on $X \times X$ is an optimal coupling of the probability measures $(e_0)_*\Pi$ and $(e_1)_*\Pi$ on X.

Here and in the sequel $e_t : \Gamma(X) \to X$ for $t \in [0, 1]$ denotes the evaluation map $\gamma \mapsto \gamma_t$. Moreover, for each measurable map $f : X \to X'$ and each measure μ on X the push forward (or image measure) of μ under f will be denoted by $f_*\mu$.

Lemma 2.2. *(i) For each pair $\mu_0, \mu_1 \in \mathcal{P}_2(X, d_X)$ there exists an optimal coupling π.*

(ii) The support of any optimal coupling π is a d_X^2-cyclically monotone set.

(iii) If X is geodesic then for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(X, d_X)$ there exists a dynamical optimal transference plan with given initial and terminal distribution: $(e_0)_*\Pi = \mu_0$ and $(e_1)_*\Pi = \mu_1$.

(iv) Given any dynamical optimal transference plan Π as above, a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}_2(X, d_X)$ connecting μ_0 and μ_1 is given by

$$\mu_t := (e_t)_*\Pi.$$

$\rightarrow$ Proofs can be found in [24],[26].

We assume that the reader is familiar with basics in Alexandrov spaces ([6], [7], [20]) and Finsler geometry ([22], [16], [5]). For a bilinear form $b : V \times V \rightarrow \mathbb{R}$ over some vector space V we will sometimes use the following abreviation $b(v, v) =: b(v)$.

2.2 Warped products of metric spaces

Let (B, d_B) and (F, d_F) be metric spaces that are complete, locally compact and intrinsic. Let $f : B \rightarrow \mathbb{R}_{\geq 0}$ be Lipschitz. Let us consider a continuous curve $\gamma = (\alpha, \beta) : [a, b] \rightarrow B \times F$. We define the length of γ by

$$\mathrm{L}(\gamma) := \sup_T \sum_{i=0}^{n-1} \left(d_B(\alpha(t_i), \alpha(t_{i+1}))^2 + f(\alpha(t_{i+1}))^2 d_F(\beta(t_i), \beta(t_{i+1}))^2\right)^{\frac{1}{2}}. \tag{3}$$

where the supremum is taken with respect to $\{(t_i)_{i=0}^n\} =: T \subset [a, b]$ with $a = t_0 < \cdots < t_n = b$. We call a curve $\gamma = (\alpha, \beta)$ in $B \times F$ admissible if α and β are Lipschitz in B and F respectively. In that case

$$\mathrm{L}(\gamma) = \int_0^1 \sqrt{v^2(t) + (f \circ \alpha)^2(t) w^2(t)} dt.$$

where v and w are the velocities of α and β in (B, d_B) and (F, d_F) respectively. For example

$$v(t) = \lim_{\epsilon \to 0} \frac{d_B(\alpha(t), \alpha(t + \epsilon))}{\epsilon}.$$

L is a length-structure on the class of admissible curves. For details see [6] and [2]. Then we can define a semi-distance between (p, x) and (q, y) by

$$\inf \mathrm{L}(\gamma) =: \big|(p, x), (q, y)\big| \in [0, \infty)$$

where the infimum is taken over all admissible curves γ that connect (p, x) and (q, y).

Definition 2.3. The warped product of metric spaces (B, d_B) and (F, d_F) with respect to a locally Lipschitz function $f : B \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$(C := B \times F/_{\sim}, |\cdot, \cdot|) =: B \times_f F$$

where the equivalence relation $\sim$ is given by

$$(p, x) \sim (q, y) \Longleftrightarrow |(p, x), (q, y)| = 0$$

and the metric distance is $|[(p, x)], [(q, y)]| := |(p, x), (q, y)|$.

Remark 2.4. One can see that

$$C = (\hat{B} \times_{\hat{f}} F) \dot\cup f^{-1}(\{0\}) \text{ where } B \backslash f^{-1}(\{0\}) =: \hat{B} \text{ and } \hat{f} = f|_{\hat{B}}.$$

We will often make use of the notation $\hat{C} := \hat{B} \times_{\hat{f}} F$. $B \times_f F$ is intrinsic. Completness and local compactness follow from the corresponding properties of B and F. It follows that $B \times_f F$ is strictly intrinsic. Especially for every pair of points we find a geodesic between them.

The next two theorems by Alexander and Bishop describe the behaviour of geodesics.

Theorem 2.5 ([2]). *For a minimizer $\gamma = (\alpha, \beta)$ in $B \times_f F$ with $f > 0$ we have*

1. *β is pregeodesic in F and has speed proportional to $f^{-2} \circ \alpha$. .*
2. *α is independent of F, except for the lengt of β.*
3. *If β is non-constant, γ has a parametrization proportional to arclength satisfying the energy equation $\frac{1}{2}v^2 + \frac{1}{2f^2} = E$ almost everywhere, where v is the speed of α and E is constant.*

Theorem 2.6 ([1]). *Let $\gamma = (\alpha, \beta)$ be a minimizer in $B \times_f F$ that intersects $X = f^{-1}(\{0\})$.*

1. *If γ has an endpoint in X, then α is a minimizer in B.*
2. *β is constant on each determinate subinterval.*
3. *α is independent of F, except for the distance between the endpoint values of β. The images of the other determinate subintervals are arbitrary.*

We want to discuss a special case in more detail.

Definition 2.7. For a geodesic metric space (M, d), the K-*cone* $(\mathrm{Con}_K(M), d_{\mathrm{Con}_K})$ is a metric space defined as follows:

- $\diamond$ $\mathrm{Con}_K(M) := \begin{cases} M \times [0, \pi/\sqrt{K}] / (M \times \{0, \pi/\sqrt{K}\}) & \text{if } K > 0 \\ M \times [0, \infty)/(M \times \{0\}) & \text{if } K \leq 0 \end{cases}$
- $\diamond$ For $(x, s), (x', t) \in \mathrm{Con}_K(M)$

$$\begin{aligned} &d_{\mathrm{Con}_K}((x, s), (x', t)) \\ &:= \begin{cases} \mathrm{cn}_K{}^{-1}\left(\mathrm{cn}_K(s)\mathrm{cn}_K(t) + K\mathrm{sn}_K(s)\mathrm{sn}_K(t)\cos\left(d(x, x') \wedge \pi\right)\right) & \text{if } K \neq 0 \\ \sqrt{s^2 + t^2 - 2st\cos\left(d(x, x') \wedge \pi\right)} & \text{if } K = 0. \end{cases} \end{aligned}$$

where $\mathrm{sn}_K(t) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}t)$ and $\mathrm{cn}_K(t) = \cos(\sqrt{K}t)$ for $K > 0$ and $\mathrm{sn}_K(t) = \frac{1}{\sqrt{-K}}\sinh(\sqrt{-K}t)$ and $\mathrm{cn}_K(t) = \cosh(\sqrt{-K}t)$ for $K < 0$.

If $\mathrm{diam}M \leq \pi$, the K-cone coincides with the warped product $[0, \infty) \times_{\mathrm{sn}_K} M$. Especially it is geodesic space.

2.3 Alexandrov spaces

In this section let B and F be finite-dimensional Alexandrov spaces with CBB by K and K_F respectively. A continuous function $f : B \to \mathbb{R}_{\geq 0}$ is said to be $\mathcal{F}K$-concave if its restriction to every unit-speed geodesic γ satisfies

$$u(t) \geq \sigma^{(1-t)}u(0) + \sigma^{(t)}u(\theta) \text{ for all } t \in [0, \theta]$$

where $\theta = \mathrm{L}(\gamma)$. For the definition of $\sigma^{(t)} = \sigma_{1,1}^{(t)}(\theta)$ see the remark directly after Definition 2.17. This is equivalent to that $u \circ \gamma$ is a sub-solution of the following Dirichlet problem

$$\begin{aligned} g'' = -Kg \quad &\text{on } (0, \theta) \\ g(0) = u \circ \lambda(0) \quad &g(\theta) = u \circ \lambda(1). \end{aligned}$$

We assume that f is locally Lipschitz, $\partial B \subseteq f^{-1}(\{0\})$ and

$$K_F \geq Kf^2(p) \quad \text{and} \quad Df_p \leq \sqrt{K_F - Kf^2(p)} \quad \forall p \in B. \tag{4}$$

Df_p is the modulus of the gradient of f at p in the sense of Alexandrov geometry (see for example [20]). In the following we will mainly stay in the setting of Riemannian manifolds, where Df_p can always be replaced by $|\nabla f_p|$.

In [1] Alexander and Bishop proved the following three results.

Proposition 2.8 ([1])**.** *For a $\mathcal{F}K$-concave function $f : B \to [0, \infty)$ on some Alexandrov space B with CBB by K, the condition (4) is equivalent to*

1. *If $X = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $X \neq \emptyset$, suppose $K_F \geq 0$ and $|\nabla f_p| \leq \sqrt{K_F}$ for all $p \in X$.*

Proposition 2.9 ([1])**.** *For f and B as in the previous proposition, assume the conditions of Proposition 2.8. Then f is positive on non-boundary points: $f^{-1}(\{0\}) \subset \partial B$.*

Proposition 2.10 ([1])**.** *Let $f : B \to \mathbb{R}_{\geq 0}$ and B as in the previous proposition. Suppose $\partial B = f^{-1}(\{0\}) \neq \emptyset$ and $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in \partial B$. Then we have: Any minimizer in $B \times_f F$ joining two points not in ∂B, and intersecting ∂B, consists of two horizontal segments whose projections to F are $\pi/\sqrt{K_F}$ apart, joined by a point in ∂B.*

The main theorem of Alexander and Bishop concerning warped products is:

Theorem 2.11 ([1])**.** *Let B and F be complete, finite-dimensional spaces with CBB by K and K_F respectively. Let $f : B \to \mathbb{R}_{\geq 0}$ be an $\mathcal{F}K$-concave, locally Lipschitz function satisfying the boundary condition (†). Set $X = f^{-1}(\{0\}) \subset \partial B$.*

1. *If $X = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $X \neq \emptyset$, suppose $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in X$.*

Then the warped product $B \times_f F$ has CBB by K.

(†) If $B^\dagger$ is the result of gluing two copies of B on the closure of the set of boundary points where f is nonvanishing, and $f^\dagger : B^\dagger \to \mathbb{R}_{\geq 0}$ is the tautological extension of f, then $B^\dagger$ has CBB by K and $f^\dagger$ is $\mathcal{F}K$-concave.

2.4 Warped products of Riemannian manifolds

We assume additionally that $B \backslash f^{-1}(\{0\}) = \hat{B}$ and F are Riemannian manifolds with dimension d and n respectively. The Riemannian warped product with respect to $\hat{f} = f|_{\hat{B}}$ is defined in the following way:

$$\hat{C} := \hat{B} \times_{\hat{f}} F := (\hat{B} \times F, g).$$

The Riemannian metric g is given by

$$g := \pi^* g_B + (f \circ \pi)^2 \pi^* g_F$$

where g_B and g_F are the Riemannian metrics of B and F respectively. It's clear that g is well-defined and Riemannian. The length of a Lipschitz-continuous curve $\gamma = (\alpha, \beta)$ in $\hat{C}$ with respect to the metric g is given by

$$\mathrm{L}(\gamma) = \int_0^1 \sqrt{g_B(\dot{\alpha}(t), \dot{\alpha}(t)) + f^2 \circ \alpha(t) g_F(\dot{\beta}(t), \dot{\beta}(t))} dt$$

So the Riemannian distance on $\hat{C}$ is defined by

$$|(p, x), (q, y)\hat{|} = \inf \mathrm{L}(\gamma)$$

where the infimum is taken over all Lipschitz curves that are joining (p,x) and (q,y) in $\hat{C}$. It is easy to see that the Riemannian warped product $\hat{C}$ as metric space isometrically embeds in the metric space warped product C and the metrics coincide on $C\backslash f^{-1}(\{0\})$. The next proposition is the smooth analogon of Theorem 2.5 and can be found in [17].

Proposition 2.12. *$\gamma=(\alpha,\beta)$ is a geodesic in $\hat{B}\times_{\hat{f}} F$ in the sense that $\nabla_{\dot\gamma}\dot\gamma=0$ if and only if*

(1) $\nabla^B_{\dot\alpha}\dot\alpha=|\dot\beta|^2(f\circ\alpha\nabla)f|_\alpha$ *in* B *(2)* $\nabla^F_{\dot\beta}\dot\beta=\frac{-2}{f\circ\alpha}(f\circ\alpha)'\dot\beta$ *in* F.

Remark 2.13. ∇^B and ∇^F denote the Levi-Civita connections of B and F respectively. Because of Theorem 2.5 we know that $|\dot\beta|^2 f^4\circ\alpha=c$ where c is constant. Then (1) becomes

$$\nabla_{\dot\alpha}\dot\alpha=-\nabla\frac{c}{2f^2}|_\alpha. \tag{5}$$

If β is constant, then $c=0$ and α is a geodesic in B. (5) also holds for a metric warped product as long as B is Riemannian.

It is possible to calculate the Ricci-tensor of $\hat{B}\times_{\hat{f}} F$ explicitly. Let $\xi+v\in T\hat{C}_{(p,x)}=TB_p\oplus TF_x$ be arbitrary and let $\tilde{X}$ and $\tilde{V}$ be horizontal and vertical lifts of vector fields X and V on $\hat{B}$ and F respectively such that $\tilde{X}_{(p,x)}+\tilde{V}_{(p,x)}=\xi+v$. For the notion of horizontal and vertical lifts we refer to [17] and we will use it without further comment. Then the formula is

$$\begin{aligned}\mathrm{ric}_{\hat{C}}(\xi+v) &= \mathrm{ric}_{\hat{C}}(\tilde{X}_{(r,x)}+\tilde{V}_{(r,x)})\\ &= \mathrm{ric}_B(X_p)-n\frac{\nabla^2 f_p(X_p)}{f(p)}+\mathrm{ric}_F(V_x)-\left(\frac{\Delta f(p)}{f(p)}+(n-1)\frac{|\nabla f|_p^2}{f^2(p)}\right)|\tilde{V}_{(p,x)}|^2\end{aligned} \tag{6}$$

and can be found in [17, chapter 7, 43 corollary]. Clearly this representation is independent from the vector fields X and V. We remark that $|V|^2=g_F(V,V)$ and $|\tilde{V}|^2=g_{\hat{C}}(\tilde{V},\tilde{V})$. $\nabla^2 f$ denotes the Hessian of f which can be defined as follows. For $v,w\in TM_p$ choose vector fields X and Y such that $X_p=v$ and $Y_p=w$. Then

$$\nabla^2 f(v,w)=X_pYf-(\nabla_XY)_pf \tag{7}$$

where ∇ denotes here the Levi-Civita-conection of g. We declare $\nabla^2 f(v)=\nabla^2 f(v,v)$. Because of (7) we will always choose vector fields in the way we did above to do calculation. The results will be independent from this choise. In this smooth setting $\mathcal{F}K$-concavity for a smooth f becomes

$$\nabla^2 f(v)\le -fK|v|^2 \text{ for any } v\in TB.$$

If we have $\hat{B}$ with $\mathrm{ric}_{\hat{B}}\ge(d-1)K$ and f $\mathcal{F}K$-concave satisfying the condition (4) or equivalently the conditions in Proposition 2.8, then

$$\mathrm{ric}_F(v)\ge(n-1)K_F|v|^2\ \ \forall v\in TF\ \Rightarrow\ \mathrm{ric}_{\hat{C}}(w)\ge(n+d-1)K|\xi+v|^2\ \ \forall(\xi+v)\in T\hat{C}.$$

2.5 Finsler manifolds

Definition 2.14. A C^∞-Finsler structure on a C^∞-manifold M is a function $\mathcal{F}:TM\to[0,\infty)$ satisfying the following conditions:

(1) (Regularity) $\mathcal{F}$ is C^∞ on $TM\backslash 0_M$, where $0_M:M\to TM$ with $0_M|_p=0\in TM_p$ denotes the zero section of TM.

(2) (Positive homogeneity) For any $v\in TM$ and positive number $\lambda>0$ we have $\mathcal{F}(\lambda v)=\lambda\mathcal{F}(v)$.

(3) (Strong convexity) Given local coordinates $(x^i, v^i)_{i=1}^n$ on $\pi^{-1}(U) \subset TM$ for $U \subset M$, then

$$(g_{i,j}(v)) := \left(\frac{1}{2} \frac{\partial^2 (\mathcal{F}^2)}{\partial v^i \partial v^j}(v) \right) \tag{8}$$

is positive-definite at every $v \in \pi^{-1}(U) \backslash 0$.

We call $(g_{i,j})_{1 \leq i,j \leq n}$ fundamental tensor and $(M, \mathcal{F})$ a Finsler manifold. $(g_{i,j})_{i,j}$ can be interpreted as Riemannian metric on the vector bundle

$$\bigcup_{v \in TM} TM_{\pi(v)} \to TM$$

that associates to every $v_p \in TM_p$ again a copy of TM_p itself. An important property of the fundamental tensor for us is its invariance under vertical rescaling:

$$g_{i,j}(v) = g_{i,j}(\lambda v) \text{ for every } \lambda > 0.$$

The Finsler structure induces a distance that makes the Finsler manifold a metric space except for the symmetry of the distance. Because we only consider symmetric metrics, we additionally assume

(4) (Symmetry) $\mathcal{F}(v) = \mathcal{F}(-v)$.

The proof of Theorem 3.8 relies on Proposition 3.3 where the symmetry of the Finsler metric doesn't play a role. Thus we expect that in the anti-symmetric case no further difficulties arise and an analogon of the main result holds true.

We need notions of curvature for a Finsler manifold $(M, \mathcal{F})$ but we want to avoid a lengthy indroduction. So we follow the lines indicated in chapter 6 in [22]. Fix a unit vector $v \in TM_p$ and extend it to a C^∞-vector field V on a neighborhoud U of p such that $V_p = v$ and every integral curve of V is a geodesic. That is always possible and we call such a vector field geodesic. Because of the strong convexity property the vector field V induces a Riemannian structur on U by

$$g_p^V := \sum_{i,j=1}^n (g_{i,j})(V_p) dx_p^i \otimes dx_p^j \text{ for all } p \in U$$

Then the *flag curvature* $\mathcal{K}(v, w)$ of v and a linearly independent vector $w \in TM_p$ is the sectional curvature of v and w with respect to g^V. Similar the *Finsler-Ricci curvature* $\mathrm{ric}(v)$ of v is the Ricci curvature of v with respect to g^V. These definitions are independent of the choise of V (see [16] and [22]).

Remark 2.15. Although we assume the squared Finsler structure $\mathcal{F}^2$ to be C^∞-smooth (what we will call just smooth) outside the zero section, the lack of regularity at 0_M is worse than one would expect. Namely $\mathcal{F}^2$ is C^2 on TM if and only if $\mathcal{F}$ is Riemannian. Otherwise we only get a regularity of order $C^{1+\alpha}$ for some $0 < \alpha < 1$. (For the statement that we have C^2 if and only if we are in a smooth Riemannian setting, see Proposition 11.3.3 in [22].) This fact has important consequences for warped products in the setting of Finsler manifolds.

Now we want to define a warped product between Finsler manifolds $(B, \mathcal{F}_B)$ and $(F, \mathcal{F}_F)$ with respect to a smooth function $f : B \to [0, \infty)$ and exactly like in the Riemannian case we can define a warped product Finsler structure explictely on $\hat{B} \times F$ by

$$\mathcal{F}_{B \times_f F} := \sqrt{\mathcal{F}_{\hat{B}}^2 \circ (\pi_B)_* + (f \circ \pi_B)^2 \mathcal{F}_F^2 \circ (\pi_F)_*}.$$

By the Remark 2.15 it is clear that $\mathcal{F}_{B \times_f F}$ is no Finsler structur on $\hat{B} \times F$ in the sense of our definition. It can't be smooth on $T\hat{B} \times 0_F$ unless F is Riemannian and analogously it can't be

smooth on $0_B \times TF$ unless B is Riemannian. Especially it is only possible to define the fundamental tensor where $\mathcal{F}_{B\times_f F}$ is smooth.

We can also consider the abstract warped product that we introduced before. Again the two definitions provide the same notion of length on $\hat{B} \times F$ and therefore they produce the same complete metric space, that we call again $B \times_f F = C$.

It will turn out that curvature bounds for B in the sense of Alexandrov are essential in our proof where we show a curvature-dimension condition for $B \times_f F$. But a general Finsler manifolds won't satisfy such a bound. So it's convenient to assume that at least $(B, \mathcal{F}_B)$ is purely Riemannian with $\mathcal{F}_B^2 = g_B$. In this case the fundamental tensor at $v_{(p,x)}$ where $\mathcal{F}_{B\times_f F}$ is smooth becomes

$$g_{i,j}(v_{(p,x)}) = \begin{cases} (g_B)_{i,j}(p) & \text{if } \ 1 \leq i,j \leq d \\ \frac{1}{2}f^2(p)\frac{\partial^2(\mathcal{F}_F^2)}{\partial v^k \partial v^l}((\pi_F)_* v_{(p,x)}) & \text{if } \ d+1 \leq i,j \leq d+n \\ 0 & \text{otherwise.} \end{cases}$$

2.6 N-Ricci curvature

Definition 2.16. Given a complete n-dimensional Riemannian manifold M equipped with its Riemannian distance d_M and weighted with a smooth measure $dm_M(x) = e^{-\Psi(x)} d\mathrm{vol}_M(x)$ for some smooth function $\Psi : M \to \mathbb{R}_{>0}$. Then for each real number $N > n$ the N-Ricci-tensor is defined as

$$\begin{aligned} \mathrm{ric}^{N,m_M}(v) := \mathrm{ric}^{N,\Psi}(v) &:= \mathrm{ric}(v) + \nabla^2\Psi(v) - \frac{1}{N-n}\nabla\Psi \otimes \nabla\Psi(v) \\ &= \mathrm{ric}(v) - (N-n)\frac{\nabla^2 \Psi^{\frac{1}{N-n}}(v)}{\Psi^{\frac{1}{N-n}}(p)} \end{aligned}$$

where $v \in TM_p$. For $N = n$ we define

$$\mathrm{ric}^{N,\Psi}(v) := \begin{cases} \mathrm{ric}(v) + \nabla^2\Psi(v) & \nabla\Psi(v) = 0 \\ -\infty & \text{else.} \end{cases}$$

For $1 \leq N < n$ we define $\mathrm{ric}^{N,\Psi}(v) := -\infty$ for all $v \neq 0$ and 0 otherwise.

We switch again to the setting of weighted Finsler manifolds $(M, \mathcal{F}_M, m_M)$. In this context the measure m_M is assumed to be smooth. That means, if we consider M in local coordinates, the measure m_M is absolutely continuous with respect to $\mathcal{L}^n$ and the density is a smooth and positive function. We remark that there is no canonical volume for Finsler manifolds which would allow us to write m_M as a density like in the Riemannian case. Motivated by the previous definition Ohta introduced in [16] for a weighted Finsler manifold the N-Ricci-tensor that we define now. For $v \in TM_p$ choose a geodesic vector field V on $U \ni p$ such that $v = V_p$. Then the geodesic $\eta : (-\epsilon, \epsilon) \to M$ with $\dot{\eta}(0) = v$ is an integral curve of V. Like above that leads to a Riemannian metric g^V on U and we have the following representation $m_M = e^{-\Psi_V} dvol_{g^V}$ on U. Then for $N \geq 1$ the N-Ricci-tensor at v is defined as

$$\mathrm{ric}^{N,m_M}(v) := \mathrm{ric}^{N,\Psi_V}(v).$$

The benefit of this definition will appear shortly after we introduced the curvature-dimension condition.

Of course in the case of our Finslerian warped product this definition makes no sense for v where the Finsler structure is not at least C^2. But if the F-component of some vector (ξ, v) is non-zero, then we will always find a corresponding geodesic vector field with non-zero F-components on some open neighborhood that makes it possible to define the N-Ricci-tensor as above.

2.7 The curvature-dimension condition

In this section we give a short survey about the curvature-dimension condition in sense of Lott-Villani/Sturm. (See [24],[25],[12].)

Definition 2.17. Let (M, d, m) be a metric measure space. Given $K \in \mathbb{R}$ and $N \in [1, \infty)$, the condition $CD(K, N)$ states that for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(M, d, m)$ there exist an optimal coupling q of $\mu_0 = \rho_0 m_M$ and $\mu_1 = \rho_1 m_M$ and a geodesic $\mu_t = \rho_t m_M$ in $\mathcal{P}_2(M, d_M, m_M)$ connecting them such that

$$\int_M \rho_t^{1-1/N'} dm \geq \int_{M\times M} \left[\tau_{K,N'}^{(1-t)}(d(x_0,x_1))\rho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}(d(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] dq(x_0,x_1)$$

for all $t \in (0, 1)$ and all $N' \geq N$.

In the case $K > 0$, the *volume distortion coefficients* $\tau_{K,N}^{(t)}(\cdot)$ for $t \in (0, 1)$ are defined by

$$\tau_{K,N}^{(t)}(\theta) = t^{1/N} \cdot \left(\sigma_{K,N-1}^{(t)}(\theta)\right)^{1-1/N} \quad \text{where } \sigma_{K,N-1}^{(t)}(\theta) = \frac{\sin\left(\sqrt{K/N-1}\theta t\right)}{\sin\left(\sqrt{K/N-1}\theta\right)}$$

if $0 \leq \theta < \sqrt{\frac{N-1}{K}}\pi$ and by $\tau_{K,N}^{(t)}(\theta) = \infty$ if $\theta \geq \sqrt{\frac{N-1}{K}}\pi$. In the case $K \leq 0$ an analogous definition applies with an appropriate replacement of $\sin\left(\sqrt{\frac{K}{N-1}}...\right)$.

The definitions of $CD(K, N)$ in [24, 25] and [12] slightly differ. For non-branching spaces, both concepts coincide. In this case, it suffices to verify (2.17) for $N' = N$ since this already implies (2.17) for all $N' \geq N$. Even more, the condition (2.17) can be formulated as a pointwise inequality.

Lemma 2.18 ([24],[25],[12])**.** *A nonbranching metric measure space (M, d_M, m_M) satisfies the curvature-dimension condition $CD(K, N)$ for given numbers K and N if and only if for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(M, d_M, m_M)$ there exist an optimal path measure ν with initial and terminal distributions $(e_0)_* = \mu_0$, $(e_1)_* = \mu_1$ such that for ν-a.e. $\gamma \in \Gamma(M)$ and all $t \in (0, 1)$*

$$\rho_t^{-1/N}(\gamma_t) \geq \tau_{K,N}^{(1-t)}(\dot\gamma) \cdot \rho_0^{-1/N}(\gamma_0) + \tau_{K,N}^{(t)}(\dot\gamma) \cdot \rho_1^{-1/N}(\gamma_1) \tag{9}$$

where $\dot\gamma := d_M(\gamma_0, \gamma_1)$ and ρ_t denotes the Radon-Nikodym density of $(e_t)_*\nu$ with respect to m_M

Theorem 2.19 ([25], [16])**.** *A weighted complete Finsler manifold without boundary $(M, \mathcal{F}_M, m_M)$ satisfies the condition $CD(K, N)$ if and only if*

$$\mathrm{ric}^{N,m_M}(v, v) \geq K\mathcal{F}_M^2(v)$$

for all $v \in TM$.

Lemma 2.20 (Generalized Bonnet-Myers Theorem, [25])**.** *Assume that a metric measure space (M, d, m) satisfies the curvature-dimension condition $CD(K, N)$ for some number $N > 1$. Then the diameter of M is bounded by $\pi\sqrt{N-1/K}$.*

Theorem 2.21 ([21])**.** *If M satisfies the curvature-dimension condition $CD(K, N)$, then M satisfies the measure contraction property (K, N)-MCP.*

A New Reference Measure for $B \times_f F$. We have to introduce a reference measure on C which reflects the warped product construction. In general we define

Definition 2.22 (N-warped product)**.** Let (B, d_B, m_B) and (F, d_F, m_F) be a metric measure spaces. For $N \in [1, \infty)$, the *N-warped product* $(C, |\cdot,\cdot|, m)$ of B and F is a metric measure space defined as follows:

- $C := B \times_f F = (B \times F/_\sim, |\cdot,\cdot|)$

- $dm_C(p,x) := \begin{cases} f^N(p)dm_B(p) \otimes dm_F(x) & \text{on } \hat{C} \\ 0 & \text{on } C\backslash\hat{C}. \end{cases}$

Remark 2.23. In the setting of K-cones we can introduce a measure in the same way. We call the resulting metric measure space (K,N)-cone.

To motivate the last definition let us consider the Riemannian case again. On the weighted Riemannian manifold F we have a function $\Psi : F \to \mathbb{R}$ and the reference measure is $e^{-\Psi} d\mathrm{vol}_F$. We don't change the volume on B because there is already the function f which can be interpreted as weight on B. The Riemannian volume of C is

$$d\mathrm{vol}_{\hat{C}} = f^n d\mathrm{vol}_{\hat{B}} d\mathrm{vol}_F$$

What is the relation to the measure introduced in previous definition? For $N \in [1,\infty)$ we have

$$dm_{\hat{C}}(p,x) = f^N(p) d\mathrm{vol}_{\hat{B}}(p) e^{-\Psi(x)} d\mathrm{vol}_F(x) = f^{N-n}(p) e^{-\Psi(x)} d\mathrm{vol}_{\hat{C}}(p,x).$$

We define a function Φ on $\hat{B}$ as follows

$$e^{-\Phi(p)} = f^{N-n}(p) \Longrightarrow \Phi(p) = -(N-n)\log f(p)$$

So our reference measure $m_{\hat{C}}$ has the density $e^{-(\Phi+\Psi)}$ with respect to $d\mathrm{vol}_C$.

3 Proof of the Main Theorem

3.1 Ricci-Tensor for Riemannian Warped Products

Proposition (3.2) is a generalisation of formula (6) for N-warped products for all $N \in [1,\infty)$, where $\hat{B} = B\backslash f^{-1}(0)$ is Riemannian and (F,Ψ) is a weighted Riemannian manifold. We remind the reader of the following proposition.

Proposition 3.1. *Consider $\xi + v \in T\hat{C}_{(p,x)} = TB_p \oplus TF_x$ and vector fields X and V with $X_p = \xi$ and $V_x = v$ and their horizontal and vertical lifts respectively. $\tilde{\nabla}$ is the Levi-Civita-conection of $\hat{B} \times_{\hat{f}} F$. Then we have on $\hat{B} \times_{\hat{f}} F$*

(i) $\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla^{\hat{B}}_X Y}$,

(ii) $\tilde{\nabla}_{\tilde{X}}\tilde{V} = \tilde{\nabla}_{\tilde{V}}\tilde{X} = \frac{Xf}{f} \circ \pi_B \tilde{V}$,

(iii) $\tilde{\nabla}_{\tilde{V}}\tilde{W} = -\frac{\langle V,W\rangle}{f} \circ \pi_B \widetilde{\nabla f} + \widetilde{\nabla^F_V W}$.

Proof. The proof is a straightforward calculation or can be found in [17]. $\square$

Proposition 3.2. *Let $\hat{B}$ be a Riemannian manifold and (F,Ψ) be a weighted Riemannian manifold. Let $N \geq 1$ and $f : \hat{B} \to (0,\infty)$ is smooth. $\hat{B} \times_f F = \hat{C}$ ist the associated N-warped product of $\hat{B}$, (F,Ψ) and f. Consider $\xi + v \in T\hat{C}_{(p,x)} = TB_p \oplus TF_x$ and vector fields X and V with $X_p = \xi$ and $V_x = v$ and their horizontal and vertical lifts $\tilde{X}$ and $\tilde{V}$ on $\hat{C}$. Then we have*

(i) $\mathrm{ric}_{\hat{C}}^{N+d,\Phi+\Psi}(\xi+v) = \mathrm{ric}_{\hat{B}}(\xi) - N\frac{\nabla^2 f(\xi)}{f(p)} + \mathrm{ric}_F^{N,\Psi}(v) - \left(\frac{\Delta f(p)}{f(p)} + (N-1)\frac{(\nabla f_p)^2}{f^2(p)}\right)|\tilde{V}_{(p,x)}|^2.$

If the reference measure on F is just the Riemannian volume and if we identify $\mathrm{ric}_F^{N,1}$ with ric_F, we especially have

(ii) $\mathrm{ric}^{N+d,\Phi}_{\hat{C}}(\xi+v) = \mathrm{ric}_{\hat{B}}(\xi) - N\frac{\nabla^2 f(\xi)}{f(p)} + \mathrm{ric}_F(v) - \left(\frac{\Delta f(p)}{f(p)} + (N-1)\frac{(\nabla f(p))^2}{f^2(p)}\right)|\tilde{V}_{(p,x)}|^2.$

We remind the reader that $|\tilde{V}_{(p,x)}|^2 = f(p)|V_x|^2$.

Proof. First we assume $N > n$ where n is the dimension of F. We can calculate the $(N+d)$-Ricci-curvature explictly by leading everything back to formula (6). We begin with the second formula. That means the reference measure on F is simply the Riemannian volume. We have to find espressions for the first and second derivative of Φ.

$$\nabla^2\Phi(\bar{W}) = \nabla^2\Phi(\tilde{X}) + \nabla^2\Phi(\tilde{V}) + 2\underbrace{\nabla^2\Phi(\tilde{X},\tilde{V})}_{=0}$$

where $\tilde{X}$ and $\tilde{V}$ are lifts of vector fields X and V. Further we have

$$\begin{aligned}
\nabla^2\Phi(\tilde{X}) &= \tilde{X}\tilde{X}\Phi - (\nabla_{\tilde{X}}\tilde{X})\Phi = \nabla^2\Phi(X) = XX\Phi - (\nabla_X X)\Phi \\
&= -(N-n)\left(X\Big(\frac{Xf}{f}\Big) - \frac{\nabla_X Xf}{f}\right) \\
&= -(N-n)\left(\Big(XXf\cdot f - Xf\cdot Xf\Big)\frac{1}{f^2} - \frac{\nabla_X Xf}{f}\right) \\
\nabla^2\Phi(\tilde{V}) &= \tilde{V}\tilde{V}\Phi - (\nabla_{\tilde{V}}\tilde{V})\Phi = -(N-n)\left(\frac{\nabla f}{f}\right)^2|\tilde{V}|^2 \\
\frac{1}{N-n}(\nabla\Phi\otimes\nabla\Phi)(\bar{W}) &= \frac{1}{N-n}(\nabla\Phi\otimes\nabla\Phi)(X) = (N-n)\frac{1}{f^2}(Xf\cdot Xf)
\end{aligned}$$

So the $(N+d)-$Ricci-curvature becomes

$$\begin{aligned}
\mathrm{ric}^{N+d,\Phi}_{\hat{C}}(\bar{W}) &= \mathrm{ric}_{\hat{C}}(\bar{W}) \\
&\quad -(N-n)\left(\frac{1}{f^2}\big(XXf\cdot f - Xf\cdot Xf\big) - \frac{\nabla_X Xf}{f} - \left(\frac{\nabla f}{f}\right)^2|\tilde{V}|^2\right) \\
&\quad -(N-n)\frac{1}{f^2}(Xf\cdot Xf) \\
&= \mathrm{ric}_{\hat{C}}(\bar{W}) \\
&\quad -(N-n)\left(\frac{1}{f^2}XXf\cdot f - \frac{\nabla_X Xf}{f} - \left(\frac{\nabla f}{f}\right)^2|\tilde{V}|^2\right) \\
&= \mathrm{ric}_{\hat{C}}(\bar{W}) \\
&\quad -(N-n)\frac{\nabla^2 f(X)}{f} - (N-n)\left(\frac{\nabla f}{f}\right)^2|\tilde{V}|^2 \\
&= \mathrm{ric}_{\hat{B}}(X) - N\frac{\nabla^2 f(X)}{f} + \mathrm{ric}_F(V) - \left(\frac{\Delta f}{f} + (N-1)\frac{|\nabla f|^2}{f^2}\right)|\tilde{V}|^2
\end{aligned}$$

Now we change the measure on F. We choose a function $\Psi : F \to \mathbb{R}$ and define the reference

measure as $e^{-\Psi} d\mathrm{vol}_F$.

$$
\begin{aligned}
\nabla^2\Psi(\bar{W}) &= \underbrace{\nabla^2\Psi(\tilde{X})}_{=0} + \nabla^2\Psi(\tilde{V}) + 2\nabla^2\Psi(\tilde{X},\tilde{V}) \\
\nabla^2\Psi(\tilde{V}) &= \tilde{V}\tilde{V}\Psi - (\nabla_{\tilde{V}}\tilde{V})\Psi \\
&= VV\Psi - (\nabla_V V)\Psi + \langle \nabla f, \nabla\Psi\rangle \frac{1}{f}|V|^2 \\
&= \nabla^2\Psi(V,V) + \langle \nabla f, \nabla\Psi\rangle \frac{1}{f}|V|^2 = \nabla^2\Psi(V,V) \\
\nabla^2\Phi(\tilde{V},\tilde{X}) &= \tilde{V}\tilde{X}\Phi - (\nabla_{\tilde{V}}\tilde{X})\Phi \\
&= -\frac{1}{f}\tilde{X}f\cdot\tilde{V}\Psi \\
(\nabla(\Psi+\Phi))\otimes(\nabla(\Psi+\Phi))(\bar{W}) &= (\nabla\Psi\otimes\nabla\Psi)(\tilde{V}) + (\nabla\Phi\otimes\nabla\Phi)(\bar{W}) + \frac{N-n}{f}\tilde{X}f\cdot\tilde{V}\Psi
\end{aligned}
$$

The $(N+d)-$Ricci-curvature becomes

$$
\begin{aligned}
\mathrm{ric}_{\hat{C}}^{N+d,\Phi+\Psi}(\bar{W}) &= \mathrm{ric}_{\hat{C}}(\bar{W}) + \nabla^2(\Phi+\Psi)(\bar{W}) \\
&\quad + \frac{1}{N-n}\left(\nabla(\Phi+\Psi)\otimes\nabla(\Phi+\Psi)(X+V)\right) \\
&= \mathrm{ric}_{\hat{C}}(X+V) + \nabla^2\Phi(X+V) + \frac{1}{N-n}(\nabla\Phi\otimes\nabla\Phi)(X+V) \\
&\quad + \nabla^2\Psi(X+V) \\
&\quad + \frac{1}{N-n}\left((\nabla\Psi\otimes\nabla\Psi)(\tilde{V}) + \frac{1}{f}\tilde{X}f\cdot\tilde{V}\Psi\right) \\
&= \mathrm{ric}_{\hat{C}}^{N+d,\Phi}(\bar{W}) + \nabla^2\Psi(X+V) \\
&\quad + \frac{1}{N-n}(\nabla\Psi\otimes\nabla\Psi)(\tilde{V}) + \frac{1}{f}\tilde{X}f\cdot\tilde{V}\Psi \\
&= \mathrm{ric}_{\hat{B}}(X) - N\frac{\nabla^2 f(X)}{f} + \mathrm{ric}_F(V) - \left(\frac{\Delta f}{f} + (N-1)\frac{\nabla f^2}{f}\right)|\tilde{V}|^2 \\
&\quad + \nabla^2\Psi(V,V) - \frac{1}{f}\tilde{X}f\cdot\tilde{V}\Psi \\
&\quad + (\nabla\Psi\otimes\nabla\Psi)(V) + \frac{1}{f}\tilde{X}f\cdot\tilde{V}\Psi \\
&= \mathrm{ric}_{\hat{B}}(X) - N\frac{\nabla^2 f(X)}{f} + \mathrm{ric}_F^{N,\Psi}(V) - \left(\frac{\Delta f}{f} + (N-1)\frac{|\nabla f|^2}{f^2}\right)|\tilde{V}|^2
\end{aligned}
$$

Now we consider the case $N=n$. $\nabla(\Psi+\Phi)(\xi+v)\neq 0$ for $\xi+v\in TC_{(p,x)}$, where $\xi\in TB_p$ and $v\in TF_x$, is equivalent to $\nabla\Psi(v)\neq 0$ or $\nabla\Phi(\xi)\neq 0$. So by definition the left hand side of our formula evalued at (ξ,v) is $=-\infty$ if and only if the right hand side is $=-\infty$.

If $\nabla(\Psi+\Phi)(\xi+v)=0$, then choose again vector fields V and X with $V_x=v$ and $X_p=\xi$, repeat the above calculation for $N>n$ and evaluate the formula at (ξ,v). The terms with $\frac{1}{N-n}$ disappear. Let $N\to n$ and get the desired result.

If $N<n$ then $\mathrm{ric}_{\hat{C}}^{N+d,\Psi+\Phi}=-\infty$ and $\mathrm{ric}_F^{N,\Psi}=-\infty$ by definition. $\square$

3.2 Finsler Case

Now we treat the case of Finsler manifolds.

Proposition 3.3. *Let (F, m_F) be a weighted Finsler manifold. $N \geq 1$, $f : \hat{B} \to (0, \infty)$ and $\hat{B}$ are as in the previous proposition. $\hat{B} \times_f F = \hat{C}$ ist the associated N-warped product. Then we have*

$$\mathrm{ric}_{\hat{C}}^{N+d,m_C}(\xi + v) = \mathrm{ric}_{\hat{B}}(\xi) - N\frac{\nabla^2 f(\xi)}{f(p)} + \mathrm{ric}_F^{N,m_F}(v) - \left(\frac{\Delta f(p)}{f(p)} + (N-1)\frac{(\nabla f_p)^2}{f^2(p)}\right)\mathcal{F}_{\hat{C}}(v)^2$$

where $\xi + v \in T\hat{C}_{(p,x)}$ with $v \neq 0$ (Especially the $N+d$-Ricci-tensor of $\hat{C}$ is well-defined at $\xi + v$ because $\mathcal{F}_{\hat{C}}$ is smooth in this direction).

Proof. Choose $\xi + v \in T\hat{C}_{(p,x)}$ such that $\mathcal{F}_{\hat{C}}(\xi + v) = 1$ and with $v \neq 0$ and a unit-speed geodesic $\gamma = (\alpha, \beta) : [-\epsilon, \epsilon] \to \hat{C}$ with $\dot{\alpha}(0) = \xi$ and $\dot{\beta}(0) = v$. We set $\gamma(-\epsilon) = (p_0, x_0)$ and $\gamma(\epsilon) = (p_1, x_1)$. We choose ϵ small such that $\mathrm{L}(\gamma) = 2\epsilon$ is sufficiently far away from the cut radius of the endpoints. Up to reparametrization β is geodesic in F between x_0 and x_1 by Theorem 2.5. Let $\bar{\beta} : [0, \mathrm{L}] \to F$ be the unit-speed reparametrization of β. That means there exists a $s : [-\epsilon, \epsilon] \to [0, L]$ such that $\bar{\beta} \circ s = \beta$. L is the length of β. There exists $t_0 \in [0, \mathrm{L}]$ such that $\bar{\beta}(t_0) = \beta(0) = x$. We have

$$\dot{\beta}(t) = s'(t)\dot{\bar{\beta}}(s(t)) = \mathcal{F}_F(\dot{\beta}(t))\dot{\bar{\beta}}(s(t))$$

We extend this last observation to the flow of geodesic vector fields $\bar{V}$ and $\bar{W}$.

$$\bar{V} = \nabla d_F(x_0, \cdot)$$

is a smooth geodesic vector field on some neighborhoud U of x. We choose U small enough such that $x_0, x_1 \notin U$ and it doesn't intersect with the cut locus of x_0. Then, if we restrict the image of $\bar{\beta}$ to U, it is an integral curve of $\bar{V}$. We can define a Riemannian metric $g^{\bar{V}}$ on U with respect to this geodesic vector field and then we can represent the measure m_F with a positive smooth density $\Psi_{\bar{V}}$ with respect to $d\mathrm{vol}_{g^{\bar{V}}}$. Additionally we have

$$\mathcal{F}_F(\lambda v) = \sqrt{g^{\bar{V}}(\lambda v, \lambda v)} \text{ and } \mathrm{ric}^{N,m_F}(\lambda v) = \mathrm{ric}^{N,\Psi_{\bar{V}}}(\lambda v) \text{ for all } \lambda > 0 \tag{10}$$

Consider the vector field

$$\bar{W} = \nabla d_{\hat{C}}((p_0, x_0), \cdot)$$

restricted to $(\hat{B} \times U) \cap \{(p, x) : (\pi_F)_*(V_p) \neq 0\} = \hat{U}$ which is open, where π_F is the projection from $\hat{C}$ to F. Every intergral curve of $\bar{W}$ coincides in $\hat{U}$ with a unit-speed geodesic from (p_0, x_0) to a point $(\bar{p}, \bar{x}) \in \hat{U}$. And especially, like we mentioned above, the projection to F of each such geodesic is after reparametrization the geodesic that connects x_0 and $\bar{x}$ in F. Thus the vertical projections of integral curves of $\bar{W}$ are integral curves of $\bar{V}$ after reparametrization. For an arbitrary integral curve $\gamma = (\alpha, \beta)$ of $\bar{W}$ we do the following explicite computation

$$\begin{aligned}
(\pi_F)_*(\bar{W}_{\gamma(t)}) &= (\pi_F)_*(\dot{\gamma}(t)) \\
&= (\pi_F)_*(\dot{\alpha}(t) + \dot{\beta}(t)) = \dot{\beta}(t) \\
&= \mathcal{F}_F(\dot{\beta}(t))\dot{\bar{\beta}}(s(t)) \\
&= \mathcal{F}_F(\dot{\beta}(t))\bar{V}_{\bar{\beta}(s(t))} = \mathcal{F}_F(\dot{\beta}(t))\bar{V}_{\beta(t)} \\
\Rightarrow \qquad (\pi_F)_*(\bar{W}_{(r,p)}) &= \mathcal{F}_F((\pi_F)_*\bar{W}_{(r,p)})\bar{V}_p \quad \forall (r, p) \in \hat{U}
\end{aligned}$$

Thus $\mathcal{F}_F((\pi_F)_*\bar{W})^{-1}\bar{W} =: W$ is π_F-related to $\bar{V}$.

We remember that by definition

$$\mathcal{F}_{\hat{C}}^2 := \mathcal{F}_{\hat{B}}^2 \circ (\pi_B)_* + (f \circ \pi)^2 \mathcal{F}_F^2 \circ (\pi_F)_*.$$

The Finslerian N-Ricci-tensor with respect to $dm_C = d\mathrm{vol}_{\hat{B}} \otimes f^N dm_F$ for vectors of $\bar{W}$ is the Riemannian N-Ricci-tensor of $g^{\bar{W}}$ with respect to $\Psi_{\bar{W}}$. The components of $g^{\bar{W}}$ are $(g_B)_{i,j}(p)$ if $1 \leq i, j \leq d$ and

$$\begin{aligned} g^{\bar{W}}_{i,j}|_{(p,x)} &= \frac{1}{2}\frac{\partial^2(\mathcal{F}^2_{\hat{C}})}{\partial v^i \partial v^j}(\bar{W}_{(p,x)}) \\ &= \frac{1}{2} f^2(p) \frac{\partial^2(\mathcal{F}^2_F)}{\partial v^i \partial v^j}((\pi_F)_* \bar{W}_{(p,x)}) \\ &= \frac{1}{2} f^2(p) \frac{\partial^2(\mathcal{F}^2_F)}{\partial v^i \partial v^j}(\mathcal{F}_F((\pi_F)_* \bar{W}_{(p,x)})\bar{V}_x) = f^2(p) g^{\bar{V}}_{i\text{-}d,j\text{-}d}|_x \end{aligned}$$

if $d+1 \leq i, j \leq n+d$. The last equality holds because the fundamental tensor is homogenous of degree zero. But then $g^{\bar{W}}|_{(p,x)} = (g_B)|_p + f^2(p)(g^{\bar{V}})|_x$ for all $(p,x) \in \hat{U}$ and we can apply the formula (6), that especially holds at $\xi + v$. Together with (10) this leads the desired formula for $\xi + v$. If $\mathcal{F}_{\hat{C}}(\xi + v) \neq 1$, consider the normalized vector, repeat everything and use (10) to get the same result. $\square$

3.3 Optimal Transportation in Warped Products

The next theorem is not tied to the context of Finsler manifolds but a purely metric space result.

Theorem 3.4. *Let (B, d_B) be a complete Alexandrov space with CBB by K and let (F, d_F, m_F) be a metric measure space satisfying the $((N-1)K_F, N)$-MCP for $N \geq 1$ and $K_F > 0$ and diam$(F) \leq \pi/\sqrt{K_F}$. Let $f : B \to \mathbb{R}_{\geq 0}$ be some $\mathcal{F}K$-concave function such that $X = \partial B = f^{-1}(\{0\})$, $X \neq \emptyset$ and*

$$K_F \geq 0 \text{ and } Df_p \leq \sqrt{K_F} \text{ for all } p \in X.$$

Consider $C = B \times_f F$. Let Π be an optimal dynamical transference plan in C such that $(e_0)_*\Pi$ is absolutely continuous with respect to m_C and $\mathsf{spt}\Pi = \Gamma$. Then the set

$$\Gamma_X := \{\gamma \in \mathsf{spt}\Pi : \exists t \in (0,1) : \gamma(t) \in X\}$$

has Π-measure 0.

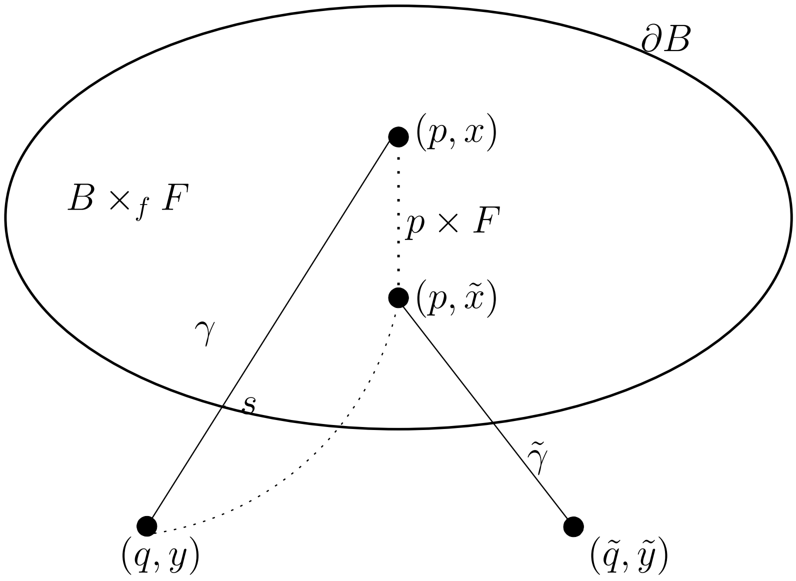
Proof. We can assume that for all $\gamma \in \Gamma$ there is a $t \in (0,1)$ such that $\gamma(t) \in X$ and without lose of generality $K_F = 1$. We set $\mu_t = (e_t)_*\Pi$ and $\mathsf{spt}\mu_t = \Omega_t$. $\pi = (e_0, e_1)_*\Pi$ is an optimal plan between μ_0 and μ_1. We assume that $\Omega_0 \cap X = \emptyset$. For the proof we use the following results of Ohta

Theorem 3.5 ([14])**.** *If a metric measure space (M, d, m) satisfies the (K, N)-MCP for some $K > 0$ and $N > 1$, then, for any $x \in M$, there exists at most one point $y \in M$ such that $|x, y| = \pi\sqrt{(N-1)/K}$.*

Lemma 3.6 ([15])**.** *Let (M, d, m) be a metric measure space satisfying the (K, N)-MCP for some $K > 0$ and $N > 1$. If diam$M = |p, q| =\pi\sqrt{\frac{N-1}{K}}$, then for every point $z \in M$, we have $|p, z| + |z, q| = |p, q|$. In particular there exists a minimal geodesic from p to q passing through z.*

We want to show that μ_0 is actually concentrated on the graph of some map $\phi : p_1(\Omega) \subset B \to F$ where $p_1 : B \times^N_f F \to B$ is the projection map. Then since the measure μ_0 is absolutely continuous with respect to the product measure $f^N d\mathcal{H}^d_B \otimes dm_F$, its total mass has to be zero by Fubinis' theorem and the fact that m_F contains no atoms. We define ϕ as follows. Choose $(p, x) \in \Omega_0$ which is starting point of some transport geodesic $\gamma = (\alpha, \beta)$. If $(p, \tilde{x}) \in \Omega_0$, we show that $x = \tilde{x}$. So ϕ can be defined by $p \mapsto x$.

Let $\gamma, \tilde{\gamma} \in \Gamma$ be transport geodesics starting in (p, x) and $(p, \tilde{x})$ respectively. For the moment we're only concerned with $\gamma = (\alpha, \beta)$. It connects (p, x) and (q, y), and since it passes through X,



by Proposition 2.10 it decomposes into $\gamma|_{[0,\tau)} = (\alpha_0, x)$, $\gamma(\tau) = s \in S$ and $\gamma|_{(\tau,1]} = (\alpha_1, y)$ where $x, y \in F$ such that $|x, y| = \pi$.

We deduce an estimate for $|(p,\tilde{x}),(q,y)|$. By Lemma 3.6 there exists a geodesic from x to y passing throug $\tilde{x}$. So by Theorem 2.5 it is enough to consider $B \times_f [0,\pi]$ instead of C. We have $x = 0$ and $y = \pi$. $(\alpha_0, \tilde{x})$ is a minimizer between $(p,\tilde{x})$ and s and especially $|s,(p,\tilde{x})| = |s,(p,x)|$.

We will essentially use a tool introduced in the proof of Proposition 7.1 in [1]. There the authors define a nonexpanding map Ψ from a section of the constant curvature space S^3_K into $B \times_f [0,\pi]$. For completeness we repeat its construction:

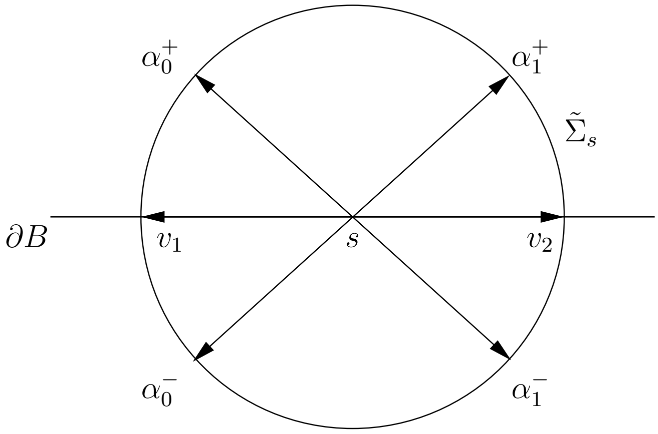
B is an Alexandrov space, so the following is well-defined. $C_K(\Sigma_s)$ denotes the K-cone over Σ_s where Σ_s denotes the space of directions of s in B. s is the point where γ intersects ∂B. So we can write down the gradient exponential map in s

$$\begin{aligned} \exp_s : C_K(\Sigma_s) &\to B \\ (t,\sigma) &\mapsto c_\sigma(t) \end{aligned}$$

where c_σ denotes the quasi-geodesic that corresponds to $\sigma \in \Sigma_s$. The gradient exponential is a generalisation of the well-known exponential map in Riemannian geometry and it is non-expanding and isometric along cone radii which correspond to minimizers in B. Quasigeodesic were introduce by Alexandrov and studied in detail by Perelman and Petrunin in [19]. $\tilde{B}$ denotes the doubling of B, that is the gluing of two copies of B along their boundaries. By a theorem of Perelman (see [18]) it is again an Alexandrov space with the same curvature bound. For s the space of direction $\tilde{\Sigma}_s$ in $\tilde{B}$ is simply the doubling of Σ_s.

We make the following observations. $\alpha_0 \star s \star \alpha_1$ has to be a geodesic in $\tilde{B}$ between p and q where p and q lie in different copies of B respectively. Otherwise there would be a shorter curve $\tilde{\alpha}_0 \star \tilde{s} \star \tilde{\alpha}_1$ that would also give a shorter path between (p,x) and (q,y) in C. We denote by α_1^+ and α_0^- the right hand side and the left hand side tangent vector at s respectively.

By reflection at ∂B we get a another curve that is again a geodesic. This curve results from α_0 and α_1 that were interpreted as curves in the other copy of B respectively. Two cases occur.



If $\alpha_0^+(t) \neq \alpha_1^-(t)$ in Σ_s then we got two pairs of directions with angle π. The case when $\alpha_0^+(t) = \alpha_1^-(t)$ will be discussed at the end. Now, in an analog way as one step before, we see that $\tilde{\Sigma}_s$ is a spherical suspension with respect to each of these pairs and that all 4 directions we consider lie on a geodesic loop $c : [0, 2\pi]/_{\{0 \sim 2\pi\}} \to \tilde{\Sigma}_s$ of length 2π. We set $\{v_1, v_2\} = \mathrm{Im} c \cap \partial\Sigma_s$. Because the second curve was obtained by reflection, clearly we have $|\alpha_0^+, v_1| = |\alpha_0^-, v_1|$ and $|\alpha_0^+, v_2| = |\alpha_0^-, v_2|$ and analogously for α_1^+ and α_1^-. So we see that there is an involutive isometry of $\mathrm{Im} c$ fixing $\{v_1, v_2\}$. But then $|v_1, v_2|$ has to be π. We use a parametrization by arclength such that $c(0) = v_1$ and $c(\pi) = v_2$ and consider $c|_{[0,\pi]} = c : [0, \pi] \to \Sigma_s$.

Now consider the space S_K^2 of dimension 2 in $\mathbb{R}^3$ and $S_K^2 \cap (\mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}) =: \hat{S}_K^2$. We introduce polar coordinates

$$\big(\mathrm{sn}_K(\varphi)\cos(\vartheta), \mathrm{sn}_K(\varphi)\sin(\vartheta), \mathrm{cn}_K(\varphi)\big) \text{ where } \vartheta \in [0, \pi] \text{ and } \varphi \in I_K := \begin{cases} [0, \pi/\sqrt{K}] & \text{if } K > 0 \\ [0, \infty) & \text{if } K \leq 0 \end{cases}$$

and the K-cone map

$$\tilde{\Psi} : \hat{S}_{K,}^2 \to C_K(\Sigma_s) \qquad \tilde{\Psi}(\varphi, \vartheta) = (\varphi, c(\vartheta))$$

which is an isometry onto $C_K(\mathrm{Im} c \cap \Sigma_s)$.

We consider $\hat{S}_K^2 \times_\Phi [0, \pi] =: \hat{S}_K^3$ where $\Phi(\varphi, \vartheta) = \sin \circ d_{\partial \hat{S}_K^2}(\varphi, \vartheta) = \mathrm{sn}_K\varphi \sin\vartheta$ and $\partial\hat{S}_K^2 = \{(\varphi, \vartheta) : \vartheta = 0 \text{ or } = \pi\} \simeq \frac{1}{\sqrt{K}} S^1$ and define the following map

$$\Psi = \exp_s \circ \tilde{\Psi} \times \mathrm{id}_{[0,\pi]} : \hat{S}_K^2 \times_\Phi [0, \pi] = \hat{S}_K^3 \to B \times_f [0, \pi]$$

From the proof of Proposition 7.1 in [1] we know that Ψ is still nonexpanding and an isometry along cone radii which correspond to minimizers in B. The essential ingredient is $\mathrm{sn}_K(\varphi) \leq f(\alpha(\varphi))$ for any geodesic α in B.

Quite similar as in the case of K-cones one can see that the distance on $\hat{S}_K^2 \times_\Phi [0, \pi]$ is explicetly given by

$$\begin{aligned} \mathrm{cn}_K|(\varphi_0, \vartheta_0, x_0), (\varphi_1, \vartheta_1, x_0)| &= \mathrm{cn}_K\varphi_0 \mathrm{cn}_K\varphi_1 \\ &\quad + K\mathrm{sn}_K\varphi_0 \cos\vartheta_0 \mathrm{sn}_K\varphi_1 \cos\vartheta_1 \\ &\quad + K\mathrm{sn}_K\varphi_0 \sin\vartheta_0 \mathrm{sn}_K\varphi_1 \sin\vartheta_1 \cos(x_0 - x_1) \\ &= \mathrm{cn}_K\varphi_0 \mathrm{cn}_K\varphi_1 \\ &\quad + K\mathrm{sn}_K\varphi_0 \mathrm{sn}_K\varphi_1 \left(\cos\vartheta_0 \cos\vartheta_1 + \sin\vartheta_0 \sin\vartheta_1 \cos(x_0 - x_1)\right) \end{aligned}$$

For $K > 0$ we deduce the desired estimate

$$\begin{aligned}
\mathrm{cn}_K|(p,\tilde{x}),(q,y)| &= \mathrm{cn}_K|\Psi((\varphi_0,\vartheta_0,\tilde{x})),\Psi((\varphi_1,\vartheta_1,y))| \\
&\geq \mathrm{cn}_K|(\varphi_0,\vartheta_0,\tilde{x})),(\varphi_1,\vartheta_1,y)| \\
&= \mathrm{cn}_K\varphi_0\mathrm{cn}_K\varphi_1 + K\mathrm{sn}_K\varphi_0\cos\vartheta_0\mathrm{sn}_K\varphi_1\cos\vartheta_1 \\
&\quad +K\mathrm{sn}_K\varphi_0\sin\vartheta_0\mathrm{sn}_K\varphi_1\sin\vartheta_1\cos(\tilde{x}-y) \\
&\geq \mathrm{cn}_K\varphi_0\mathrm{cn}_K\varphi_1 + K\mathrm{sn}_K\varphi_0\mathrm{sn}_K\varphi_1(\cos\vartheta_0\cos\vartheta_1 - \sin\vartheta_0\sin\vartheta_1) \\
&= \mathrm{cn}_K\varphi_0\mathrm{cn}_K\varphi_1 + K\mathrm{sn}_K\varphi_0\mathrm{sn}_K\varphi_1\cos(\vartheta_0+\vartheta_1) \\
&\geq \mathrm{cn}_K\varphi_0\mathrm{cn}_K\varphi_1 - K\mathrm{sn}_K\varphi_0\mathrm{sn}_K\varphi_1 \\
&= \mathrm{cn}_K(\varphi_0+\varphi_1) \\
&= \mathrm{cn}_K(|s,(p,\tilde{x})| + |s,(q,y)|) \\
&= \mathrm{cn}_K(|s,(p,x)| + |s,(q,y)|) = \mathrm{cn}_K(|(p,x),(q,y)|)
\end{aligned}$$

$$\Longrightarrow \quad |(p,\tilde{x}),(q,y)| \leq |(p,x),(q,y)| \tag{11}$$

with equality in the second inequality if and only if $|y,\tilde{x}| = \pi$. The case $K \leq 0$ follows in the same way but we have to be aware of reversed inequalities and minus signs that will appear. We get the same estimate for (p,x) and $(\tilde{q},\tilde{y})$. By optimality of the plan we have

$$|(p,\tilde{x}),(q,y)|^2 + |(p,x),(\tilde{q},\tilde{y})|^2 \geq |(p,x),(q,y)|^2 + |(p,\tilde{x}),(\tilde{q},\tilde{y})|^2$$

and from that we have equality in (11). So we got $|x,\tilde{y}| = \pi$ and $|y,\tilde{x}| = \pi$. But by Ohta's theorem antipodes are unique and thus we got $y = \tilde{y}$ and $x = \tilde{x}$.

The case when $\alpha_0^+(t) = \alpha_1^-(t)$ works as follows. The last identity implies w.l.o.g. $\mathrm{Im}\alpha_1 \subset \mathrm{Im}\alpha_0$. We define a map from the K-cone into the warped product

$$\hat{\Psi} : I_K \times_{\mathrm{sn}_K} [0,\pi] \to B \times_f [0,\pi] \text{ by } (\varphi,x) \mapsto (\alpha_0(\varphi),x).$$

Again $\hat{\Psi}$ is nonexpanding. By following the lines of Bacher/Sturm in [3] we get the same estimate as in (11). □

Existence of optimal maps. We have already mentioned that the Finsler structure on $\hat{C}$ is not smooth, or more precisely $\mathcal{F}_C^2$ is C^1 but not C^2 at any $v \in T\hat{B}_p \oplus O_F$. So we cannot apply the classical existence theorem for optimal maps. But the special situation of warped products allows to proof the existence of optimal maps by following the lines given in chapter 10 of [26]. There the cost function comes from a Lagrangian that living on a Riemannian manifold. It is easy to see that the Riemannian structure is not so important. But the Lagrangian viewpoint fits perfectly well to our setting if we consider $L : T\hat{C} \to \mathbb{R}$ with $L(v) = \mathcal{F}_C^2(v)$. The associated action functional is

$$\mathcal{A}(\gamma) = \int_0^1 \mathcal{F}_C^2(\dot{\gamma}(t))dt$$

where $\gamma : [0,1] \to \hat{C}$ is a Lipschitz curve. Minimizers of this action functional are just the constant speed geodesics of $\hat{C}$. We have the following theorem.

Theorem 3.7. *Given $\mu,\nu \in \mathcal{P}^2(\hat{C})$ that are compactly supported and such that μ is absolutely continous with respect to m_C. Take compact sets $Y \supset \operatorname{supp}\nu$ and $X = \bar{U}$ such that $\operatorname{supp}\mu \subset U$. Then there exists a $\frac{1}{2}d^2$-concave function $\phi : X \to \mathbb{R}_{\geq 0}$ relative to (X,Y) such that the following holds: $\pi = (Id_{\hat{C}},T)_*\mu$ is a unique optimal coupling of (μ,ν), where $T : X \to Y$ is a measurable map and defined m_C-almost everywhere by $T((p,x)) = \gamma^{(p,x)}(1)$ where $\gamma^{(p,x)}$ is a constant-speed geodesic and uniquely determined by $-d\phi_{(p,x)}(\dot{\gamma}^{(p,x)}(0)) = \mathcal{F}^2(\dot{\gamma}^{(p,x)}(0))$.*

For completeness we give a self-contained presentation of the proof from [26] in the Appendix where our discussion closely follows [13] and [16].

3.4 Proof of the Main Theorem

Theorem 3.8. *Let B be a complete, d-dimensional space with CBB by K such that $B\backslash\partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K$-concave and smooth on $B\backslash\partial B$. Assume $\partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N \geq 1$ and $K_F \in \mathbb{R}$. If $N = 1$ and $K_F > 0$, we assume that* $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. *In any case F satisfies $CD((N-1)K_F, N)$ where $K_F \in \mathbb{R}$ such that*

1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $|\nabla f|_p \leq \sqrt{K_F}$ for all $p \in \partial B$.*

Then the N-warped product $B \times_f^N F$ satisfies $CD((N + d - 1)K, N + d)$.

Proof. Let $\partial B \neq \emptyset$. For non-constant f we have $K_F > 0$. Otherwise the warped product is just the ordinary Euclidean product and the result is the tensorization property of the CD-condition (see [10]). In the case of $N > 1$ the curvature-dimension condition for (F, d_F, m_F) implies $((N-1)K_F, N)$-MCP ([21]). If $N = 1$, then by assumption we have $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. So in any case Theorem 3.4 yields that positive mass will never transported trough the set of singularity points S. So one could think to apply Theorem 2.19 to get the result because on $\hat{B} \times_{\hat{f}}^N F$ the N-Ricci-tensor is bounded in the correct way by Proposition 3.2 and our assumptions.

Two problems occur. First, the warped product without its singularity points is not geodesically complete. But if we consider some displacement interpolation between bounded and absolutely continous measures in $\hat{B} \times_{\hat{f}}^N F$ then as we have seen the transport geodesics don't intersect S. So by truncation we can find an ε-environment of the singularity set such that the transport takes place in the complement of this environment and the exceptional mass can be chosen arbitrarily small. Then in the case where F is Riemannian the calculus that was introduced in [9] is available like in the complete setting and one gets the convexity of the Jacobian of the optimal map along the transport geodesics which leads to the curvature-dimension condition (see also [3]). When $\partial B = \emptyset$ this step is redundant because no singularity points appear.

Second, if F is Finslerian, the warped product structure is not smooth on $T\hat{B} \times O_F$. So we cannot follow the lines of [16] as we did with [9] in the Riemannian case. But we know, if $\gamma = (\alpha, \beta)$ is a geodesic in $\hat{B} \times_{\hat{f}}^N F$ then by Theorem 2.5 β is a pre-geodesic. So either β is constant and α is a geodesic in B, or there exists a strictly monotone reparametrization s such that $\bar{\beta} = \beta \circ s$ is a constant speed geodesic in F. We use this fact to circumvent the problem that comes from the non-smoothness. The idea is to split the intial measure of some optimal mass transportation in $\hat{B} \times_{\hat{f}}^N F$ in two disjoint parts that will follow one of these two kinds of geodesics either. To do so we need that a point $(p, x) \in \operatorname{supp} \mu_0$ already determines the transport geodesic that starts in (p, x) uniquely. But this follows from the existence of an optimal map.

So we proceed as follows. Let μ_0 and μ_1 be absolutely continuous probability measure in $\hat{C}$. We assume wlog that μ_0 and μ_1 are compactly supported. Otherwise, we have to choose compact exhaustions of $\hat{C} \times \hat{C}$ and to consider the restriction of the plan to these compact sets. For this we also refer to [23, Lemma 3.1]. By Theorem 3.7 there is an unique optimal map $T : X \to Y$ between μ_0 and μ_1. So the unique optimal plan is given by $(\mathrm{id}, T)_*\mu_0 = \pi$ and the associated optimal dynamical plan is given by $\gamma_*\mu_0 = \Pi$ where $\gamma : \operatorname{supp} \mu_0 \to \mathcal{G}(\hat{C})$ with $(p, x) \mapsto \gamma^{(p,x)}$. The geodesic in $\mathcal{P}^2(\hat{C})$ with respect to L^2-Wasserstein distance is given by $\mu_t = (\gamma_t^{(p,x)})_*\mu_0$. We have

$$\operatorname{supp} \Gamma = \underbrace{\left\{\gamma : \dot{\gamma} \in T\hat{B} \times 0_F\right\}}_{=:\Gamma_a} \dot{\cup} \underbrace{\left\{\gamma : \dot{\gamma} \in T\hat{B} \times TF\backslash 0_F\right\}}_{=:\Gamma_b}.$$

We set $\Pi(\Gamma_a)^{-1}\Pi|_{\Gamma_a} =: \Pi_a$ and $\Pi(\Gamma_b)^{-1}\Pi|_{\Gamma_b} = \Pi_b$ that are again optimal dynamical plans. The corresponding L^2-Wasserstein geodesics are $(e_t)_*\Pi_a = \mu_{a,t}$ and $(e_t)_*\Pi_b = \mu_{b,t}$. They are absolutely continuous with densities $\rho_{a,t}$ and $\rho_{b,t}$ and have disjoint support for any $t \in [0, 1]$ because of the

optimal map and since $\hat{C}$ is non-branching (see [4, Lemma 2.6]). We have for any $t \in [0,1]$

$$\rho_t dm_C = \mu_t = \Pi(\Gamma_a)\mu_{a,t} + \Pi(\Gamma_b)\mu_{b,t} = \Pi(\Gamma_a)\rho_{a,t} dm_C + \Pi(\Gamma_b)\rho_{b,t} dm_C.$$

So the Rényi entropy functional from Definition 2.17 splits for any $t \in [0,1]$

$$\int_M \rho_t^{1-1/N'} dm_C = \Pi(\Gamma_a)^{1-1/N'} \int_M \rho_{a,t}^{1-1/N'} dm_C + \Pi(\Gamma_b)^{1-1/N'} \int_M \rho_{b,t}^{1-1/N'} dm_C$$

for any $N' \geq N$. So it suffices to show displacement convexity along Π_a and Π_b separately.

We begin with Π_a. We can approximate Π_a in L^2-Wasserstein distance arbitrarily close by

$$\frac{1}{n} \sum_{i=1}^{n} \Pi_{a,B}^i \otimes \nu_i$$

where $\Pi_{a,B}^i$ are geometric optimal transference plans in (B, d_B) and ν_i are disjoint absolutely continuous propability measures in F. So it suffices to show displacement convexity along $\Pi_{a,B}^i$. But since B has CBB by K and f is $\mathcal{F}K$-concave, $(B, d_B, f^N d\mathrm{vol}_B)$ satisfies $CD((N+d-1)K, N+d)$ (see [25, Theorem 1.7]) and the desired convexity in $\Pi_{a,B}$ follows at once.

Now consider Π_b. We know a priori that the transport geodesics only follow smooth directions of the Finslerian warped product structure. So we can consider $\mathcal{F}_C^2$ restricted to $T\hat{B} \times TF \backslash 0_F$. We get the exponential map on $T\hat{B} \times TF\backslash 0_F$ and we also can define the Legendre transformation, that yields gradient vector fields. Especially, if we consider an optimal transport that follows only smooth direction, the techniques from [16] can be applied. Thus there exists an optimal map T_b of the form $T_b((p,x)) = \exp(-\nabla\phi_{(p,x)})$ for some c-concave function ϕ. To make this more precise we can consider the complement of an ϵ-neighbourhood $\mathcal{U}_\epsilon$ of $T\hat{B} \times 0_F$ and restrict the intial measure $\mu_{b,0}$ of Π_b to the set

$$U_\epsilon = \left\{ (p,x) \in \operatorname{supp} \mu_{b,0} : \dot{\gamma}^{(p,x)}(0) \notin \mathcal{U}_\epsilon \right\}.$$

U_ϵ is measurable because the mapping $(x,p) \in\mapsto \dot{\gamma}^{(p,x)}$ is measurable. Again the exceptional mass can be chosen arbitrarily small. The optimal map T, which has been derived in Theorem 3.7, restricted to U_ϵ has to coincide with T_b because optimality is stable under restriction and because of uniqueness of optimal maps. Especially we can deduce $\mu_{b,t} = (T_t)_* \mu_{b,0}$ where $T_t((p,x)) = \exp(-t\nabla\phi_{(p,x)})$. Again by results from [16] we know ϕ is second order differentiable at least on U_ϵ. Hence the Jacobian of T_t exists and satisfies because of Proposition 3.3 and our assumptions the correct convexity condition. $\square$

Corollary 3.9. *Let B be a complete, d-dimensional space with CBB by K such that $B\backslash\partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K$-concave and smooth on $B\backslash\partial B$. Assume $\emptyset \neq \partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N > 1$. Then the following statements are equivalent*

(i) (F, m_F) satisfies $CD((N-1)K_F, N)$ with $K_F \geq 0$ and

$$|\nabla f|_p \leq \sqrt{K_F} \text{ for all } p \in \partial B.$$

(ii) The N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$

Proof. Only one direction is left. Assume the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$. B has CBB by K in the sense of Alexandrov and f is $\mathcal{F}K$-concave. Proposition 3.3 says that

$$(N+d-1)K|X|^2 \leq \mathrm{ric}_{\hat{B}}(X) - N\frac{\nabla^2 f(X)}{f} \tag{12}$$

$$(N+d-1)K\mathcal{F}^2_{\hat{B}\times_{\hat{f}}F}(\tilde{V}) \leq \mathrm{ric}_F^{N,m_F}(V) - \left(\frac{\Delta f}{f} + (N-1)\frac{\nabla f^2}{f^2}\right)\mathcal{F}^2_{\hat{B}\times_{\hat{f}}F}(\tilde{V}) \tag{13}$$

The assumptions on B and f and also (12) imply that $(B, f^N \mathrm{dvol}_B)$ satisfies $CD((N+d-1)K, N+d)$. From (13) we get

$$\mathrm{ric}_F^{N,m_F}(V) \geq (N-1)|\nabla f|_p^2 \mathcal{F}_F^2(V)$$

for all $p \in \partial B$ and all $V \in TF$. The last inequality implies that $|\nabla f|^2$ is bounded on ∂B. So we will find some $K_F \geq 0$ such that

$$|\nabla f|_p^2 \leq K_F \quad \forall p \in \partial B \tag{14}$$

So the N-Ricci-tensor of F is bounded by K_F that satisfies desired property. □

Corollary 3.10. *Let B be a complete, d-dimensional space with CBB by K such that $B \backslash \partial B$ is a Riemannian manifold. Let $f : B \to \mathbb{R}_{\geq 0}$ a function such that it is smooth and satisfies $\nabla^2 f = -Kf$ on $B \backslash \partial B$. Assume $\partial B \subseteq f^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N > 1$. Then the following statements are equivalent*

(i) (F, m_F) satisfies $CD((N-1)K_F, N)$ with $K_F \in \mathbb{R}$ such that

1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*
2. *If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $|\nabla f|_p \leq \sqrt{K_F}$ for all $p \in \partial B$.*

(ii) The N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$

Proof. Only one direction is left and we only need to consider the case wher $\partial B = \emptyset$. Assume the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$. Like in the proof of previous corollary we can deduce (12) and (13). From (13) and from the assumption on f we get

$$\mathrm{ric}_F^{N,m_F}(V) \geq (N-1)\left(Kf^2(p) + |\nabla f|_p^2\right) \mathcal{F}_F^2(V)$$

for all $p \in B$ and all $V \in TF$. The last inequality implies that $|\nabla f|^2 + Kf^2$ is bounded on B. So we will find some minimal $K_F \in \mathbb{R}$ such that

$$K_F \geq Kf^2(p) \quad \text{and} \quad |\nabla f|_p^2 + Kf^2(p) \leq K_F \quad \forall p \in B \tag{15}$$

By Proposition 2.8 this is equivalent to conditions 1. and 2. in the theorem and the N-Ricci-tensor of F is bounded by K_F. □

Remark 3.11. Like in the theorem of Alexander and Bishop our result can be extended to the case where B satisfies the boundary condition (†).

Remark 3.12. If $B = [0, \pi/\sqrt{K}]$ and $f = \mathrm{sn}_K$ (with appropriate interpretation if $K \leq 0$), the associated warped products are K-cones. If F is a Riemannian manifold in this setting we get the theorem of Sturm and Bacher from [3]. However, if F is Finslerian, the result is new.

Corollary 3.13. *For any real number $N > 1$, $CD(N-1, N)$ for a weighted Finsler manifold is equivalent to $CD(K \cdot N, N+1)$ for the associated (K, N)-cone.*

Remark 3.14. Theorem 3.4 is true when B is an Alexandrov space and F some general metric measure space. So it is reasonable to assume that our main result also could hold in a non-smooth setting and we conjecture the following

Conjecture 3.15. *Let (B, d_B) be a complete Alexandrov-space with $\dim_B = d$ and let (F, d_F, m_F) be a metric measure space. Let $f : B \to \mathbb{R}_{\geq 0}$ be some continuous function such that $\partial B \subset f^{-1}(\{0\})$. Assume that (F, m_F) satisfies $CD((N-1)K_F, N)$ and f is $\mathcal{F}K$-concave such that*

1. *If $\partial B = \emptyset$, suppose $K_F \geq Kf^2$.*

2. If $\partial B \neq \emptyset$, suppose $K_F \geq 0$ and $Df_p \leq \sqrt{K_F}$ for all $p \in X$.

Then the N-warped product $B \times_f^N F$ satisfies $CD((N+d-1)K, N+d)$

Remark 3.16. In [3] there is an example where the euclidean cone over some Riemannian manifold with $CD(N-1, N)$ produces a metric measure space satisfying $CD(0, N+1)$ but that is not an Alexandrov space with curvature bounded from below. They consider $F = \frac{1}{\sqrt{3}}S^2 \times \frac{1}{\sqrt{3}}S^2$ which satisfies $CD(3,4)$ but has sectional curvature 0 for planes spanned by vectors that lie in different spheres. Then the sectional curvature bound for the cone explodes when one gets nearer and nearer to the apex. For general warped products the same phenomenon occurs what can be seen at once from the formula of sectional curvature for warped products. Choose any closed n-dimensional Riemannian manifold with Ricci curvature bounded from below by $(n-1)K_F$ and with sectional curvature $K_F(V_x, W_x) = 0$ for some vectors V_x and W_x in $TF|_x$ (for example choose $\lambda S^m \times \lambda S^m$ where $m + m = n$ and λ is an appropriate scaling factor, that produces the Ricci curvature bound $(n-1)K_F$). Let B be a Riemannian manifold with boundary and sectional curvature bigger than $K \in \mathbb{R}$ and f is $\mathcal{F}K$-concave and satisfies the assumption of the theorem. (for example choose B as the upper hemisphere of S^d and f as the first nontrivial eigenfunction of the laplacian of this sphere. Especially f vanishes at the boundary of B and $|\nabla f|_{\partial B} = 1$. See [8]). The sectional curvature of the plane $\Pi_{(p,x)}$ spanned by vectors $(X_p, V_x), (Y_p, W_x)$ in $T(B \times_f F)_{(p,x)}$ is

$$\begin{aligned}
K(\Pi_{(x,p)}) &= K_B(X_p, Y_p)|X_p|^2|Y_p|^2 - f(p)\left[|W_x|^2\nabla^2 f(X_p, X_p) + |W_x|^2\nabla^2 f(Y_p, Y_p)\right] \\
&\quad + \frac{1}{f^2(p)}\left[K_F(V_x, W_x) - |\nabla f_p|^2\right]|\tilde{V}_x|^2|\tilde{W}_x|^2 \\
&= K_B(X_p, Y_p)|X_p|^2|Y_p|^2 - f(p)\left[|W_x|^2\nabla^2 f(X_p, X_p) + |W_x|^2\nabla^2 f(Y_p, Y_p)\right] \\
&\quad - \frac{1}{f^2(p)}|\nabla f_p|^2|\tilde{V}_x|^2|\tilde{W}_x|^2.
\end{aligned}$$

Hence the sectional curvature of planes $\Pi_{(p_n,x)} \subset T(\hat{B} \times_{\hat{f}}^N F)_{(p_n,x)}$ as above explodes to $-\infty$ if we choose a sequences (X_{p_n}) and (Y_{p_n}) such that p_n tends to vanishing points of f. On the other hand the Ricci-curvature is still bounded by 0 by formula (6). Especially there is also no upper bound for the sectional curvature.

Remark 3.17. In (3.2) and (3.3) there appears $\mathrm{ric}_{\hat{B}} - N\frac{\nabla^2 f}{f}$ which is actually the $N+d$-Ricci-tensor for the weighted Riemannian manifold $(B, g_B, f^N d\mathrm{vol}_B)$. So one could think that we could weaken the assumption on B and f from bounds on sectional curvature and $\mathcal{F}K$-concavity respectively to a curvature-dimension condition for $(B, g_B, f^N d\mathrm{vol}_B)$. On the other hand the proofs of Theorem 3.4 and Proposition 2.8 use the curvature bound in the sense of Alexandrov for B in a very essential way. Especially for Proposition 2.8 a curvature bound K in the sense of Alexandrov for B is needed and this propostion is the key that connects the formula in (3.2) with Theorem 2.10. So it seems convenient to assume that B is an Alexandrov space with CBB by K where K then also appears as $\mathcal{F}K$-concavity of f.

Proposition 3.18. *Let (B, d_B, m_B) and (F, d_F, m_F) be metric measure spaces and let $f : B \to \mathbb{R}_{\geq 0}$ locally Lipschitz. For technical reasons we assume that B is compact and small balls in F are geodesically convex. Assume the N-warped product $B \times_f^N F$ satisfies $CD(\kappa, d)$. Then $(B, d_B, f^N dm_B)$ satisfies $CD(\kappa, d)$ as well.*

Proof. Let $x \in F$ and consider $B \times_f^N B_\epsilon(x)$. Since $B_\epsilon(x)$ is convex, $B \times_f^N B_\epsilon(x) \subset B \times_f^N F$ is convex as well. Measure and distance coincide with the induced ones. So $B \times_f^N B_\epsilon(x)$ inherits the curvature-dimension bound of the ambient space. Additionally it is quite easy to see that $B \times_f^N B_\epsilon(x)$ converge to $(B, d_B, f^N dm_B)$ in sense of measured Gromov-Hausdorff convergence when $\epsilon \to 0$. But since the curvature-dimension condition is stable under measured Gromov-Hausdorff convergence, the result follows. □

Another application of Theorem 3.4 is the following result which modifies a theorem by Lott that was proven in [11].

Corollary 3.19. *Let (B, g) be a compact, n-dimensional Riemannian manifold with distance function d_B. Let $f : B \to \mathbb{R}_{\geq 0}$ be a smooth function with geodesically convex support such that $(\operatorname{supp} f, d, f dvol_B)$ satisfies $CD(\kappa, N)$ for $N \in \mathbb{N}$. Assume that for $K_F > 0$*

$$|\nabla f^{\frac{1}{q}}|_p^2 \leq K_F \quad \forall p \in \partial \operatorname{supp} f \text{ where } q = N - n. \tag{16}$$

Then $(\operatorname{supp} f, d_B, f dvol_B)$ is the Gromov-Hausdorff limit of a sequence of compact geodesic spaces (M_i, d_i) of Hausdorff dimension N satisfying $CD(\kappa, N)$.

Proof. Just follow the proof of Lott in [11] and add the main theorem. □

We have the Conjecture 3.15 but at the moment we are not able to proof it. But one could ask if it is true when F is a warped product itself and satisfies a curvature dimension bound in the sense of our main theorem. In this situation F would not be a manifold and singularities would occur. However the proof of the following corollary shows that an iterated warped product is essentially again a simple warped product.

Corollary 3.20. *Let B_2 be complete, d_2-dimensional space with CBB by K_2 such that $B_2 \backslash \partial B_2$ is a Riemannian manifold and let $f_2 : B_2 \to \mathbb{R}_{\geq 0}$ be $\mathcal{F}K_2$-concave and smooth on $B_2 \backslash \partial B_2$. Assume $\emptyset \neq \partial B_2 \subseteq f_2^{-1}(\{0\})$. Let B_1 be complete, d_1-dimensional Riemannian manifold with CBB by K_1 where $K_1 \geq 0$ such that*

$$|\nabla f_2|_p \leq \sqrt{K_1} \text{ for all } p \in \partial B_2.$$

Let $f_1 : B_1 \to \mathbb{R}_{\geq 0}$ be a smooth and $\mathcal{F}K_1$-concave. Assume $\emptyset \neq \partial B \subseteq f_1^{-1}(\{0\})$. Let (F, m_F) be a weighted, complete Finsler manifold. Let $N \geq 1$ and $K_F \in \mathbb{R}$. If $N = 1$ and $K_F > 0$, we assume that $\operatorname{diam} F \leq \pi/\sqrt{K_F}$. In any case F satisfies $CD((N-1)K_F, N)$ where $K_F \geq$ such that

$$|\nabla f_1|_p \leq \sqrt{K_F} \text{ for all } p \in \partial B_1.$$

Then the $N + d_1$-warped product $B_2 \times_{f_2}^{N+d_1} \left(B_1 \times_{f_1}^N F\right)$ satisfies $CD((N + d_1 + d_2)K_2, N + d_1 + d_2)$.

Proof. First we see that

$$B_2 \times_{f_2}^{N+d_1} \left(B_1 \times_{f_1}^N F\right) = \left(B_2 \times_{f_2}^{d_1} B_1\right) \times_{f_2 f_1}^N F.$$

as metric measure spaces. This comes from the fact that the warped product measure in both cases is

$$f_2^{N+d_1} \mathrm{dvol}_{B_2} \otimes \left(f_1^N \mathrm{dvol}_{B_1} \otimes dm_F\right) = (f_1 f_2)^N \left(f_2^{d_1} \mathrm{dvol}_{B_2} \otimes \mathrm{dvol}_{B_1}\right) \otimes dm_F$$

and the warped product metrics coincide because in both cases the length structure is given by

$$\mathrm{L}(\gamma) = \int_0^1 \sqrt{|\dot{\alpha_2}(t)|^2 + f_2^2 \circ \alpha_2(t) |\dot{\alpha}(t)_1|^2 + f_2^2 \circ \alpha_2(t) f_1^2 \circ \alpha_1(t) \mathcal{F}_F^2(\dot{\beta}(t))} dt.$$

Hence it is enough to check that $(B_2 \times_{f_2}^{d_1} B_1) \times_{f_2 f_1}^N F$ satisfies the required curvature dimension bound.

We know by Theorem 2.11 that $B_2 \times_{f_2} B_1 =: B$ is a space with CBB by K_2. It is easy to see that its boundary is $B_2 \times_{f_2} \partial B_1 = \partial B$ and that the singularity points ∂B_2 are a subset of ∂B. It follows that $B \backslash \partial B$ is a Riemannian manifold. Then we know that if $(p_2, p_1) \in \partial B$, we have $f_2(p_2) f_1(p_1) = 0$ and so $\partial B \subset f^{-1}(\{0\})$ where $f = f_2 f_1$. Then we can calculate that f is $\mathcal{F}K_2$-concave and that it satisfies

$$|\nabla f|_{(p_2, p_1)} \leq \sqrt{K_F} \quad \text{for all } (p_2, p_1) \in \partial B,$$

where the modulus of the gradient is taken with respect to the warped product metric of $B_2 \times_{f_2}^{d_1} B_1$. Thus the assumptions of Theorem 3.8 are fullfilled and the result follows. □

A Appendix

Existence of optimal maps.

Proposition A.1. *For any $(p,x) \in \hat{C}$ and any $\xi + v \in T\hat{C}_{(p,x)}$ there is a unique geodesic γ starting in (p,x) with initial tangent vector $\dot{\gamma}(0) = \xi + v$.*

Proof. If $v = 0$, $\gamma(t) = (\alpha(t), \beta(0))$ is a geodesic in B and hence uniquely determined by $\dot{\alpha}(0)$. Otherwise we have $\mathcal{F}_F^2(\dot{\beta}) f^4(\alpha) = const =: c$ (see Theorem 2.5) and α is determined by

$$\nabla_{\dot{\alpha}} \dot{\alpha} = -\nabla \tfrac{c}{2f^2}|_{\alpha}$$

and $\alpha(0)$ and $\dot{\alpha}(0)$. Together with the uniqueness property of geodesics in F, the statement follows. $\square$

For the rest of this section c stands for the cost function $c((p,x),(q,y)) = \frac{1}{2}|(p,x),(q,y)|^2 = \frac{1}{2}\inf \mathcal{A}(\gamma)$ where the infimum is taken over Lipschitz curves connecting (p,x) and (q,y). We need some background information on c-concave functions where we also refer to [13].

Definition A.2. Let $X, Y \subset \hat{C}$ be compact. Given an arbitrary function $\phi : X \to \mathbb{R} \cup \{-\infty\}$, its c-transform $\phi^c : Y \to \mathbb{R} \cup \{-\infty\}$ relative to (X,Y) is defined by

$$\phi^c((q,y)) := \inf_{(p,x)\in X} \{c((p,x),(q,y)) - \phi((p,x))\}.$$

Similar we define the c-transform of a function $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ relative to (Y,X). A function $\phi : X \to \mathbb{R} \cup \{-\infty\}$ is said to be c-concave relative to (X,Y) if it is not identical $-\infty$ and if there is a function $\psi : Y \to \mathbb{R} \cup \{\infty\}$ such that $\psi^c = \phi$.

Lemma A.3. *If ϕ is c-concave relative to (X,Y), then it is Lipschitz continuous with respect to $|\cdot,\cdot|$ and its Lipschitz constant is bounded above by some constant depending only on X and Y.*

Remark A.4. Since a c-concave function is Lipschitz continuous, it is differentiable almost everywhere. We also have that $d\phi : \hat{C} \to T^*\hat{C}$ is measurable (see [13, Lemma 4]).

Definition A.5. Let M be a manifold and $f : M \to \mathbb{R}$ be a function. A co-vector $\alpha \in T^*M_x$ is called subgradient of f at x if we have

$$f(\sigma(1)) \geq f(\sigma(0)) + \alpha(\dot{\sigma}(0)) + o(\mathcal{F}(\dot{\sigma}(0))$$

for any geodesic $\sigma : [0,1] \to M$ with $\sigma(0) = x$. The set of subgradients at x is denoted by $\partial_-^* f(x)$. Analogously we can define the set $\partial_+^* f(x)$ of supergradients at x.

Remark A.6. If f admits a sub- and supergradient at x, it is differentiable at x and $\partial_-^* f(x) = \partial_+^* f(x) = \{df_x\}$ ([26, Proposition 10.7]).

Proposition A.7. *Suppose $\gamma : [0,1] \to \hat{C}$ is a constant speed geodesic joining (p,x) and (q,y). Then $f(\cdot) = c(\cdot,(q,y))$ has supergradient $-d_v \mathcal{F}_{\hat{C}}^2|_{\dot{\gamma}(0)} \in T^*\hat{C}_{\gamma(0)}$ at (p,x) where*

$$d_v \mathcal{F}_{\dot{\gamma}(0)}^2(w) = \frac{d}{dt}\mathcal{F}^2(\dot{\gamma}(0) + tw) \quad \textit{for } w \in T\hat{C}|_{\gamma(0)}$$

Proof. Let $(\tilde{p},\tilde{x})$ and $(\tilde{q},\tilde{y})$ are points that are very close to (p,x) and (q,y) such that there are unique geodesics $\sigma_0, \sigma_1 : [0,1] \to \hat{C}$ between (p,x) and $(\tilde{p},\tilde{x})$ and between (q,y) and $(\tilde{q},\tilde{y})$ respectively. Let $\tilde{\gamma}$ be an arbitrary curve that connects $(\tilde{p},\tilde{x})$ and $(\tilde{q},\tilde{y})$. Then we have by the formula of first variation

$$\int_0^1 \mathcal{F}_C^2(\dot{\tilde{\gamma}}(t))dt = \int_0^1 \mathcal{F}_C^2(\dot{\gamma}(t))dt + d_v\mathcal{F}_C^2|_{\dot{\gamma}(1)}(\dot{\sigma}_1(0)) - d_v\mathcal{F}_C^2|_{\dot{\gamma}(0)}(\dot{\sigma}_0(0)) + o(\sup_{t\in[0,1]}|\gamma(t),\tilde{\gamma}(t)|).$$

Hence we can proof for some $\tilde{\gamma}$ with $(\tilde{q}, \tilde{y}) = (q, y)$ that

$$c((\tilde{p}, \tilde{x}), (q, y)) \leq \int_0^1 \mathcal{F}_C^2(\dot{\tilde{\gamma}}(t))dt \leq c((p, x), (q, y)) - d_v \mathcal{F}_C^2|_{\dot{\gamma}(0)}(\dot{\sigma}_0(0)) + o(|(p, x), (\tilde{p}, \tilde{x})|)$$

which means that $c(\cdot, (q, y))$ has supergradient $-d_v \mathcal{F}_C^2|_{\dot{\gamma}(0)}$. For more details we refer to [26, Proposition 10.15]. ☐

Lemma A.8. *Let $X, Y \subset \hat{C}$ be two compact subsets and $\phi : X \to \mathbb{R}$ be a c-concave function. If ϕ is differentiable in $(p, x) \in X$, and*

$$c((p, x), (q, y)) = \phi((p, x)) + \phi^c((q, y)), \tag{17}$$

then there is a geodesic $\gamma = \gamma^{(p,x)}$ between (p, x) and (q, y) satisfying $-d\phi_{(p,x)}(\dot{\gamma}(0)) = \mathcal{F}^2(\dot{\gamma}(0))$. The point (q, y) and the geodesic γ are uniquely determined by (p, x) and ϕ.

Proof. By definition of c-concave functions we have $\geq$ in (17) for any pair of points. Now choose (p, x) and (q, y) such that (17) holds and ϕ is differentiable at (p, x). Then we have for any $(\tilde{p}, \tilde{x})$

$$\phi((\tilde{p}, \tilde{x})) - \phi((p, x)) \leq c((\tilde{p}, \tilde{x}, (q, y)) - c((p, x), (q, y))$$

Instead of the point $(\tilde{p}, \tilde{x})$ we insert a curve $\sigma : (0, \epsilon) \to X$ (parametrized by arclength). Then we deduce

$$-d\phi_{(p,x)}(\dot{\sigma}) = -\frac{d}{d\epsilon}\phi \circ \sigma|_{\epsilon=0} \leq \liminf_{\epsilon \to 0} \frac{c(\sigma(\epsilon), (q, y)) - c((p, x), (q, y))}{\epsilon}$$

It follows that $d\phi_{(p,x)}$ is a subgradient of $c(\cdot, (q, y))$ at (p, x). But by the previous proposition $c(\cdot, (q, y))$ has also a supergradient at (p, x). Thus it is differentiable at (p, x) with

$$d_v \mathcal{F}_{\hat{C}}^2|_{\dot{\gamma}(0)} = dc(\cdot, (q, y))_{(p,x)} = -d\phi_{(p,x)} \tag{18}$$

where γ is some geodesic that connects (p, x) and (q, y). Now we know that $\mathcal{F}_C^2$ is strictly convex in v and C^1. Thus the co-vector $d_v \mathcal{F}_{\hat{C}}^2|_{\dot{\gamma}(0)}$ determines $\dot{\gamma}(0)$ uniquely by $d_v \mathcal{F}_{\hat{C}}^2|_{\dot{\gamma}(0)}(w) = \mathcal{F}^2(w)$ and therefore γ by Proposition A.1. So (18) and the strict convexity of $\mathcal{F}_C^2$ with respect to v determines y uniquely. ☐

Remark A.9. On $T\hat{B} \oplus TF \backslash 0_F$ we have that $\dot{\gamma}^{(p,x)}$ coincides with the gradient of $-\phi$ at x, that can be defined via Legendre transformation, and $\gamma^{(p,x)}(t) = \exp(-t\nabla\phi_{(p,x)})$. On $T\hat{B} \oplus 0_F$ it coincides with the gradient that comes from the Riemannian structure on B. The map $(p, x) \mapsto \dot{\gamma}^{(p,x)}$ is measurable because ϕ is Lipschitz (see Remark A.4) and because the transformation $\alpha \in T^*C_{(x,p)} \mapsto \alpha^* \in TC_{(p,x)}$ is continuous where α^* is uniquely determined by $\alpha(\alpha^*) = \mathcal{F}^2(\alpha^*)$. Now one can deduce that also $(p, x) \mapsto \gamma^{(p,x)}(1)$ is measurable by considering the "exponential map" separately on $T\hat{B} \oplus TF \backslash 0_F$ and $T\hat{B} \oplus 0_F$. In [26, Theorem 10.28] measurability of T is deduced by applying a measurable selection theorem.

Proof of Theorem 3.7. Let π be an optimal transference plan. By Kantorovich duality there exists a c-concave function ϕ such that $\phi((p, x)) + \phi^c((q, y)) \leq c((p, x), (q, y))$ everywhere on $\operatorname{supp} \pi \subset X \times Y$, with equality π-almost surely. Since ϕ is differentiable m_C-almost surely and since μ is absolutely continuous with respect to m_C, we can define T by Lemma A.8 m_C-almost surely by $T((p, x)) = \gamma^{(p,x)}(1)$ where $\gamma^{(p,x)}$ is uniquely given by $-d\phi_{(p,x)}(\dot{\gamma}(0)) = \mathcal{F}^2(\dot{\gamma}(0))$. Thus π is concentrated on the graph of T, or equivalently $\pi = (Id_{\hat{C}}, T)_*\mu$. Now from [16, Lemma 4.9] we know that in our setting $-d\phi$ is unique among all maximizers (ϕ, ϕ^c) of Kantorovich duality as long as the initial measure is absolutely continuous. So also T and π are unique. ☐

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